EXPLICIT PLANCHEREL FORMULA

RUI CHEN

1. Introduction

In this note, I will summarize the main result for the paper [Rap21] where he proved an explicit Plancherel formula for $GL_n(F)\backslash GL_n(E)$ where E/F is a quadratic extension of local fields of characteristic 0, the main ingredient in his proof is a local functional equation for Asai γ-factors.

2. Preliminaries

2.1. **Local γ-factors.** Let $\varphi:\ W'_F \to GL(M)$ be a continuous semi-stable and algebraic when restricted to $SL_2(\mathbb{C})$ is a finite dimensional complex representation of W'_F. We can associate φ a local L-factor $L(s,\varphi)$ and a local ϵ-factor $\epsilon(s,\varphi,\psi')$ as in Tate thesis. When $\varphi = 1_F$ is the trivial one-dimensional representation of W'_F, we will write $\zeta_F(s)$ for $L(s,1_F)$.

Definition 2.1. We define the local γ-factor associated to φ as

$$\gamma(s,\varphi,\psi') = \epsilon(s,\varphi,\psi')\frac{L(1-s,\varphi^\vee)}{L(s,\varphi)}$$

for tempered φ meaning that $\varphi(W_F)$ is bounded in $GL(M)$, we will set

$$\gamma^*(0,\varphi,\psi') = \lim_{s\to 0}\zeta_F(s)^{n_\varphi}\gamma(s,\varphi,\psi')$$

where n_φ is the order of the zero of $\gamma(s,\varphi,\psi')$ at $s=0$.

For every $\pi \in \Pi_2(GL_n(E))$, $\gamma(s,\pi,As)$ has at most a simple zero at $s=0$ and moreover

$$\gamma(0,\pi,As) \leftrightarrow \pi \in BC_n(\mathrm{Temp}(U(n)))$$

we have the following equality

$$\gamma(s,\sigma,Ad) = \frac{\gamma(s,BC_n(\sigma),Ad)}{\gamma(s,BC_n(\sigma),As)}$$

for $\sigma \in \mathrm{Temp}(U(V))$ and $n = \dim\,(V)$.

Consider the semi-direct product

$$H = (GL(M)\times GL(M)) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts by permuting the two factors of $GL(M)$, this is the L-group of $\mathrm{Res}_{E/F}(GL(V))$ with $\dim_E V = \dim M$. The irreducible representation $M\boxtimes M$ of $H^0 = GL(M)\times GL(M)$ is invariant under $\mathbb{Z}/2\mathbb{Z}$ and has two extensions to H. In one such extension, the group $\mathbb{Z}/2\mathbb{Z}$ acts by permuting the two copies of M, the other extension is given by twisting the non-trivial character of H/H^0, we will denote these extensions by $As^+(M)$ and $As^-(M)$, we have

$$\mathrm{Ind}_{H^0}^H(M\boxtimes M) = As^+(M)\oplus As^-(M)$$

Proposition 2.2. *If dim $M = n$, then the stabilizer in H of a non-degenerate vector in $As^{(-1)^{n-1}}(M)$ is isomorphic as a complex Lie group to the L-group of $U(V)$, and the action of this stabilizer on the other representation $As^{(-1)^n}(M)$ is the adjoint representation of ${}^LU(V)$.*

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For σ a unramified representation of the unitary group U_2 associated with a unramified quadratic extension E/F, assume that $\sigma \cong \mathrm{Ind}_B^G(\chi)$, with χ a unrmified character of $E^\times$, then we have

$$L_F(s,\sigma,Ad) = \zeta_F(s) L_F(s,\chi_{E/F}) L_F(s,\chi) L_F(s,\chi^{-1})$$

on the other hand for $\pi = BC(\sigma)$, we have $\pi \cong \mathrm{Ind}_{B_E}^{G_E}(\chi,\chi^{-1})$, we have the following formula for the Asai L-function for π

$$L_{AS}(s,\pi) = L_F(s,\chi) L_F(s,\chi^{-1}) \zeta_E(s)$$

hence we have the equality

$$L_{AS}(s, BC(\sigma)) = L_F(s,\sigma)$$

2.2. **Rankin-Selberg local functional equation for Asai γ-factors.** The following variants of the local Rankin-Selberg zeta integrals has been introduced and studied by Flicker and Kable, for every $W \in C^\omega(N_n(E)\backslash GL_n(E), \psi_n)$, $\phi \in S(F^n)$ and $s \in \mathcal{H}$ set

$$Z(s,W,\phi) := \int_{N_n(F)\backslash GL_n(F)} W(h)\phi(e_n h)|det\ h|^s\ dh$$

Theorem 2.3. *For every $W \in \mathcal{W}(\pi,\psi_n)$ and $\phi \in S(F^n)$ and $s \in \mathcal{H}$ with $R(s) < 1$ we have*

$$Z(1-s,\widetilde{W},\hat{\phi}) = \omega_\pi(\tau)^{n-1}|\tau|_E^{\frac{n(n-1)}{2}(s-1/2)} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \gamma(s,\pi,As,\psi') Z(s,W,\phi)$$

Lemma 2.4. *Let $\sigma \in Temp(U(n))$ and set $\pi = BC_n(\sigma)$, then we have*

$$Z(1,\widetilde{W},\hat{\phi}) = \phi(0) c_1(\pi) \beta(W)$$

for all $W \in \mathcal{W}(\pi,\psi_n)$ and $\phi \in S(F^n)$.

2.3. **Computation of certain spectral distributions.**

Proposition 2.5. *For every $\Phi \in S(Temp(\overline{GL_n(E)}))$, we have*

$$\lim_{s\to 0^+} n\gamma(s,1_F,\psi') \int_{Temp(\overline{GL_n(E)})} \Phi(\pi)\gamma(s,\pi,As,\psi')^{-1} d_{\overline{GL_n(E)}}(\pi)$$

$$\lambda_{E/F}(\psi')^{-n^2} \int_{Temp(U(n))/stab} \Phi(BC_n(\sigma)) \frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}\ d\sigma$$

where the right-hand side is absolutely convergent and so does the left hand side.

For every $\sigma \in Temp(U(n))/stab$, set

$$c(\sigma) := \lambda_{E/F}(\psi')^{-n(n+1)/2} c_1(\pi) \omega_\sigma(-1)^{1-n} \eta_{E/F}(-1)^{n(n-1)^2/2}$$

where $\pi = BC_n(\sigma)$ and $c_1(\pi)$ is a constant, note that $c(\sigma)$ is just certain root of unity.

We can obtain the following corollary from the previous proposition 2.5 and the functional equation for the zeta integral associated with the Asai L-function:

Corollary 2.6. *For every $f \in S(GL_n(E))$ and $g \in GL_n(E)$, we have*

$$\int_{N_n(F)\backslash GL_n(F)} W_f(g,h)\ dh =$$

$$|\tau|_E^{n(n-1)/4} \int_{Temp(U(n))/stab} \beta(W_{f,BC_n(\sigma)}(g,\cdot)) \frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}\ c(\sigma) d\sigma$$

Proof. We may assume that $g = 1$ by replacing f by $L(g)f$. Applying the functional equation of theorem 2.3, this becomes

$$\int_{N_n(F)\backslash GL_n(F)} W_f(1,h)dh = \lim_{s\to 0^+} n\gamma(s,1_F,\psi')|\tau|_E^{\frac{n(n-1)}{2}(\frac{1}{2}-s)} \lambda_{E/F}(\psi')^{\frac{n(n-1)}{2}} \times$$

$$\int_{Temp(\overline{GL_n(E)})} Z(1-s,\widetilde{W_{f,\pi}}(1,\cdot),\hat{\phi}) \omega_\pi(\tau)^{1-n} \gamma(s,\pi,As,\psi')^{-1} d_{\overline{GL_n(E)}}(\pi)$$

by taking the limit to 0, we get

$$\lambda_{E/F}(\psi')^{-\frac{n(n+1)}{2}}|\tau|_E^{\frac{n(n-1)}{4}}\int_{Temp(U(n))/stab} Z(1,\widetilde{W}_{f,BC_n(\sigma)}(1,\cdot),\hat{\phi})\omega_{BC_n(\sigma)}(\tau)^{1-n}\frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}\ d\sigma$$

by lemma 2.4 and the fact that $\phi(0)=1$, we have

$$Z(1,\widetilde{W}_{f,BC_n(\sigma)}(1,\cdot),\hat{\phi}) = c_1(\pi)\beta(W_{f,BC_n(\sigma)}(1,\cdot))$$

this gives the result we want. □

3. A Plancherel formula for $GL_n(F)\backslash GL_n(E)$

3.1. **A local unfolding identity.** Recall that we have associated to any $f\in S(GL_n(E))$ a function $W_f \in C^\omega(N_n(E)\backslash GL_n(E)\times N_n(E)\backslash GL_n(E), \psi_n^{-1}\boxtimes\psi_n)$.

Proposition 3.1. *For every $f\in S(GL_n(E))$, we have*

$$\int_{GL_n(F)} f(h)dh = |\tau|_E^{n(n-1)/4}\int_{N_n(F)\backslash P_n(F)}\int_{N_n(F)\backslash GL_n(F)} W_f(g,h)\ dhdp$$

where the right hand side is given by an absolutely convergent expression.

3.2. **Main theorem.**

Lemma 3.2. *For $f_1,f_2\in S(GL_n(E))$, we have*

$$(f_1,f_2)_{Y_n,\pi} = |\tau|_E^{n(n-1)/2}(\beta\hat{\otimes}\beta)(W_{\overline{f_2}*f_1^\vee,\pi})$$

Theorem 3.3. *For every $\varphi_1,\ \varphi_2\in S(Y_n)$, we have*

$$(\varphi_1,\varphi_2)_{Y_n} = \int_{Temp(U(n))/stab}(\varphi_1,\varphi_2)_{Y_n,BC_n(\sigma)}\frac{|\gamma^*(0,\sigma,Ad,\psi')|}{|S_\sigma|}\ d\sigma$$

where the right-hand side is absolutely convergent.

I will summarize BP's proof and I will ignore all the issues concerning the analytic property of the integrals.

Proof. Let $\varphi_1,\ \varphi_2\in S(Y_n)$ and choose $f_1,f_2\in S(GL_n(E))$ such that $\varphi_i=\varphi_{f_i}$, then we have

$$(\varphi_1,\varphi_2)_{Y_n} = \int_{Y_n}\int_{GL_n(F)\times GL_n(F)} f_1(h_1x)\overline{f_2(h_2x)}dh_1dh_2dx = \int_{GL_n(F)} f(h)\ dh \tag{3.1}$$

where we have set $f=\overline{f_2}*f_1^\vee$. Moreover by lemma 3.2, we also have

$$(\varphi_1,\varphi_2)_{Y_n,\pi} = |\tau|_E^{n(n-1)/2}(\beta\hat{\otimes}\beta)(W_{f,\pi}) \tag{3.2}$$

for every $\pi\in BC_n(Temp(U(n)))$.

Let $\varphi_1,\varphi_2\in S(Y_n)$ and $f_1,f_2,f\in S(GL_n(E))$ as before, by proposition 3.1 and corollary 2.6 we have

$$\int_{GL_n(F)} f(h)dh = |\tau|_E^{n(n-1)/2}\int_{N_n(F)\backslash P_n(F)}\int_{Temp(U(n))/stab}\beta(W_{f,BC_n(\sigma)(p,\cdot)})\frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}\ d\sigma\ dp$$

There are various argument in BP's paper tell us that the right hand side is an absolutely convergent expression and therefore

$$\begin{aligned}\int_{GL_n(F)} f(h)dh &= |\tau|_E^{n(n-1)/2}\int_{N_n(F)\backslash P_n(F)}\int_{Temp(U(n))/stab}\beta(W_{f,BC_n(\sigma)(p,\cdot)})\frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}\ d\sigma\ dp\\ &= |\tau|_E^{n(n-1)/2}\int_{Temp(U(n))/stab}(\beta\hat{\otimes}\beta)(W_{f,BC_n(\sigma)})\frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}c(\sigma)\ d\sigma\\ &= \int_{Temp(U(n))/stab}(\varphi_1,\varphi_2)_{Y_n,BC(\sigma)}\frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}c(\sigma)d\sigma\end{aligned}$$

By equation 3.1, we get

$$(\varphi_1,\varphi_2)_{Y_n} = \int_{Temp(U(n))/stab}(\varphi_1,\varphi_2)_{Y_n,BC(\sigma)}\frac{\gamma^*(0,\sigma,Ad,\psi')}{|S_\sigma|}c(\sigma)d\sigma$$

□

References

[Rap21] Beuzart-Plessis Raphaël. Plancherel formula for $GL_n(F)\backslash GL_n(E)$ and applications to the Ichino–Ikeda and formal degree conjectures for unitary groups. *Inventiones mathematicae*, 225(1):159–297, 2021.