EXPLICIT PLANCHEREL FORMULA FOR THE SPACE OF SYMPLECTIC FORMS

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1. Introduction

This is a study note for Lapid and Offen's proof [LO22] on the explicit Plancherel formula for the space Y_n of symplectic bilinear forms on a $2n$-dimensional vector space over a local field F in terms of the Plancherel formula of $G' = \mathrm{GL}_n(F)$.

For any $\pi \in \mathrm{Irr}_{temp}\ G'$ and $\sigma = \mathfrak{S}(\pi) \in \mathrm{Irr}\ G$, we will define a realization $\mathfrak{M}_{\psi_N}(\sigma)$ of σ with an explicit invariant inner product and a non-trivial H-invariant linear form ℓ_H.

For any $f_1, f_2 \in \mathcal{S}(G)$ let

$$(f_1, f_2)_\sigma = \sum_v \ell_H(\sigma(f_1)v)\overline{\ell_H(\sigma(f_2)v)}$$

where v ranges over a suitable orthonormal basis of $\mathfrak{M}_{\psi_N}(\sigma)$. Since ℓ_H is H-invariant, the positive semi-definite hermitian form $(f_1, f_2)_\sigma$ factors through the canonical map $\mathcal{S}(G) \to \mathcal{S}(H\backslash G)$.

Our main result is

Theorem 1.1. *For a suitable choice of Haar measures and for any $\varphi_1,\ \varphi_2 \in \mathcal{S}(H\backslash G)$ we have*

$$(\varphi_1, \varphi_2)_{L^2(H\backslash G)} = \int_{Irr_{temp}(G')} (\varphi_1, \varphi_2)_{\mathfrak{S}(\pi)}\ d\mu_{pl}(\pi)$$

where the right handside is an absolutely convergent integral.

Corollary 1.2. *We have the following decomposition of unitary representations of G:*

$$L^2(H\backslash G) = \int_{Irr_{temp}(G')} \mathfrak{S}(\pi)\ d\mu_{pl}(\pi)$$

The proof of theorem 1.1 are modeled in the recent remarkable paper of Beuzart-Plessis. The main new input is an identity described in theorem 2.7 below. It is based on two ingredients, the first is a relation between the inner product on $\mathfrak{M}_{\psi_N}(\sigma)$ and the standard invariant pairing between $I_P(\pi\otimes\pi, \varpi)$ and $I_P(\tilde{\pi}\otimes\tilde{\pi}, \varpi^{-1})$. The second is a relation between the ℓ_H and an H-invariant functional on $I_P(\pi\otimes\pi, \varpi)$ which is a local analog of the global symplectic period studied by Jacquet and Rallis. Both relations involve the standard intertwining operator from $I_P(\pi\otimes\pi, \varpi)$ to $I_P(\pi\otimes\pi, \varpi^{-1})$.

2. An identity between Bessel distributions

Let V and $V^\vee$ be two admissible smooth representations of an ℓ-group X and let

$$B : V \times V^\vee \longrightarrow \mathbb{C}$$

be an X-invariant non-degenerate bilinear form, thus B defines an isomorphism between $V^\vee$ and the smooth dual of V. We refer to $(V, V^\vee, B)$ with the group action as X-representations in duality.

A morphism

$$(\Phi, \Phi^\vee) :\ (V_1, V_1^\vee, B_1) \longrightarrow (V_2, V_2^\vee, B_2)$$

of X-representations in duality is a pair of intertwining operators $\Phi :\ V_1 \to V_2$, $\Phi^\vee :\ V_2^\vee \to V_1^\vee$ such that

$$B_2(\Phi v_1, v_2^\vee) = B_1(v_1, \Phi^\vee v_2^\vee)$$

for all $v_1 \in V_1,\ v_2 \in V_2^\vee$.

We will consider the following examples associated to an irreducible tempered representation π of $G' = GL_n(F)$.

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Example 2.1. Let $\mathfrak{M}_{\psi_{N'}}(\pi)$ be the Whittaker model of π with respect to $\psi_{N'}$ with right translation. Define a bilinear form

$$B_{Q'}(W,W^\vee)=\int_{N'\backslash Q'} W(g)W^\vee(g)\ dg$$

on $\mathfrak{M}_{\psi_{N'}}(\pi)\times\mathfrak{M}_{\psi_{N'}^{-1}}(\tilde{\pi})$, this integral is absolutely convergent by a well-known resuslt of Bernstein. Thus the tuple

$$\mathcal{D}_{\psi_{N'}}(\pi)=(\mathfrak{M}_{\psi_{N'}}(\pi),\mathfrak{M}_{\psi_{N'}}(\tilde{\pi}),B_{Q'})$$

is a G' representation in duality.

Example 2.2. Let $\mathcal{D}=(V,V^\vee,B)$ be M-representations in duality, denote the representation on V by τ, for any character χ of M consider the induced representation $I_P(V,\chi)$ realized in the space of functions $\varphi:\ G\to V$. Then we have a G-duality data

$$I_P(\mathcal{D},\chi)=(I_P(V,\chi),I_P(V^\vee,\chi^{-1}),B_{P\backslash G})$$

where

$$B_{P\backslash G}(\varphi,\varphi^\vee)=\int_{P\backslash G} B(\varphi(g),\varphi^\vee(g))\ dg$$

Example 2.3. Let $\sigma\in\mathfrak{S}(\pi)\in\mathrm{Irr}\ G$, note that $\tilde{\sigma}=\mathfrak{S}(\tilde{\pi})$, consider the Zelevinsky model $\mathfrak{M}_{\psi_N}(\sigma)$ of σ, denote D the joint stabilizer of e_n and e_{2n}, then the bilinear form

$$B_D(W,W^\vee)=\int_{D\cap N\backslash D} W(g)W^\vee(g)\ dg$$

on $\mathfrak{M}_{\psi_N}(\sigma)\times\mathfrak{M}_{\psi_N^{-1}}(\tilde{\sigma})$ is G-invariant. We write

$$\mathcal{D}_{\psi_N}(\sigma)=(\mathfrak{M}_{\psi_N}(\sigma),\mathfrak{M}_{\psi_N^{-1}}(\tilde{\sigma}),B_D)$$

for the G-representations in duality.

Let

$$\mathcal{M}_\pi:\ I_P(\mathfrak{M}_{\psi_{N_M}}(\pi\otimes\pi),\varpi)\to I_P(\mathfrak{M}_{\psi_{N_M}}(\pi\otimes\pi),\varpi^{-1})$$

be the intertwining operator given by the absolutely convergent integral

$$[\mathcal{M}_\pi\varphi(g)](x)=\int_U \varphi(\omega_U^{-1}ug)(\omega_U^{-1}x\omega_U)\ du,\ g\in G,\ x\in M$$

let $\tilde{\mathcal{M}_\pi}$ be the composition of $\mathcal{M}_\pi$ with injective map $I_P(\mathfrak{M}_{\psi_{N_M}}(\pi\otimes\pi),\varpi^{-1})\to C^\infty(N\backslash G,\psi_N)$ given by $\varphi\mapsto\varphi(g)(e)$, the image of $\tilde{\mathcal{M}_\pi}$ is $\mathfrak{M}_{\psi_N}(\sigma)$.

Definition 2.4. Suppose that $\mathcal{D}=(V,V^\vee,B)$ is an admissible X-representation in duality, denote by π the corresponding representation of X on V, using B we may identify $V\otimes V^\vee$ with the space $\mathrm{End}_{sm}(V)$ of smooth linear endomorphisms of V, i.e. the linear maps $A:V\to V$ such that $A\pi(g)=\pi(g)A=A$ for all g in sufficiently small open subgroup of X. For any $f\in\mathcal{S}(X)$, we may view $\pi(f)$ as an element of $V\otimes V^\vee$. For any $\ell\in V^*$ and $\ell^\vee\in(V^\vee)^*$, we define

$$\mathcal{B}_{\mathcal{D}}^{\ell,\ell^\vee}(f)=(\ell\otimes\ell^\vee)[\pi(f)]$$

Definition 2.5. Let $\mathfrak{B}$ be a basis for V and let K be an open subgroup of X, we say that $\mathfrak{B}$ is compatible with K if for every $v\in\mathfrak{B}$, either v is K-invariant or $\int_K\ \pi(k)v\ dk=0$. We say that $\mathfrak{B}$ is *admissible* if it is compatible with a family of open subgroups of X that form a neighborhood base for the identity.

Lemma 2.6. *We have*

- *Suppose $\mathfrak{B}$ is an admissible basis for V and let $\mathfrak{B}^\vee$ be the dual basis for $V^\vee$, then*

$$\mathcal{B}_{\mathcal{D}}^{\ell,\ell^\vee}(f)=\sum_{v\in\mathfrak{B}}\ell(\pi(f)v)\ell^\vee(\tilde{v})$$

  where only finitely many terms in the sum are non-zero.
- *For any $g_1,g_2\in X$, we have '*

$$\mathcal{B}_{\mathcal{D}}^{\ell,\ell^\vee}(L(g_1)R(g_2)f)=\mathcal{B}_{\mathcal{D}}^{\ell\circ\pi(g_1),\ell^\vee\circ\pi^\vee(g_2)}(f)$$

We now state the main result of this section. Let $\pi \in \mathrm{Irr}_{temp}(G')$ and ℓ_H be the linear form on $\mathfrak{M}_{\psi_N}(\mathfrak{S}(\pi))$ given by

$$\ell_H(W) = \int_{N_H \backslash Q_H} W(h)\ dh$$

Theorem 2.7. *Let $\pi \in Irr_{temp}(G')$ and $\sigma = \mathfrak{S}(\pi)$, then we have*

$$\mathcal{B}^{\ell_H,\delta_e}_{\mathcal{D}_{\psi_N}(\sigma)}(f) = \mathcal{B}^{\delta_e,\delta_e}_{\mathcal{D}_{\psi_{N'}}(\pi)}(Tf),\ \ f \in \mathcal{S}(G)$$

where δ_e is the evaluation at the identity and $T(f) \in \mathcal{S}(G')$ is given by

$$Tf(g) = |det\ g|^{\frac{1-n}{2}} \int_{U_H \backslash U} \varphi(u\iota_1(g))\ du, \quad g \in G'$$

where $\varphi = \int_H\ f(h\cdot)\ dh$.

A concrete M_H-invariant linear form on $\mathfrak{M}_{\psi_{N_M}}(\pi \otimes \pi)$ is given by

$$\ell_{M_H}(W) = \int_{N' \backslash G'} W(\iota(p))\ dp$$

hence the linear form

$$\ell_H^{\mathrm{ind}}(\varphi) = \int_{P_H \backslash H} \ell_{M_H}(\varphi(h))\ dh$$

$\varphi \in I_P(\mathfrak{M}_{\psi_{N_M}}(\pi \otimes \pi), \varpi)$ is well-defined and H-invariant.

Proposition 2.8. *We have*

$$\ell_H^{ind} = \ell_H \circ \tilde{\mathcal{M}}_\pi$$

Finally

Lemma 2.9. *For any $f \in \mathcal{S}(M)$ we have*

$$\mathcal{B}^{\ell_{M_H},\delta_e}_{\mathcal{D}_{\psi_{N_M}}(\pi\otimes\pi)}(f) = \mathcal{B}^{\delta_e,\delta_e}_{\mathcal{D}_{\psi_{N'}}(\pi)}(\mathfrak{C}(f))$$

where $\mathfrak{C}(f) \in \mathcal{S}(G')$ is given by

$$\mathfrak{C}(f)(g) = \int_{M_H} f(m\iota_1(g))\ dm,\ \ g \in G'$$

3. Completion of proof

The first step is take $\varphi_i = \int_H\ f_i(h\cdot)\ dh\ i = 1, 2$, the relation we wants to prove becomes

$$\int_H f(h)\ dh = \int_{\mathrm{Irr}_{temp}(G')} \mathcal{B}^{\ell_H,\ell_H}_{\mathcal{D}_{\psi_N}(\mathfrak{S}(\pi))}(f^\vee)\ d\mu_{pl}(\pi) \tag{3.1}$$

where $f = \overline{f_2} * f_1^\vee$.

We define

$$W_f(g_1, g_2) = \int_N f(g_1^{-1}ug_2)\psi_N(u)^{-1}\ du = \int_{N_M}\int_U f(g_1^{-1}vug_2)\ du\ \psi_{N_M}(v)^{-1}\ g_1,\ g_2 \in G$$

We have the following unfolding lemma

Lemma 3.1. *For any $f \in \mathcal{S}(G)$ we have*

$$\int_H f(h)\ dh = \int_{N_H \backslash Q_H} \int_{N_H \backslash H} W_f(h, q)\ dh\ dq$$

where the right-hand side converges as an iterated integral.

Next we recall the Whittaker spectral expansion for G'

Proposition 3.2. *For any $f \in \mathcal{S}(G')$*

$$W_f^{G'}(e,e) = \int_{Irr_{temp}(G')} \mathcal{B}^{\delta_e,\delta_e}_{\mathcal{D}_{\psi_{N'}}(\pi)}(f^\vee)\ d\mu_{pl}(\pi) = \int_{Irr_{temp}(G')} \mathcal{B}^{\delta_e,\delta_e}_{\mathcal{D}_{\psi^{-1}_{N'}}(\pi)}(f)\ d\mu_{pl}(\pi)$$

Now let's turn to the proof of our main theorem 1.1: combining lemma 3.1 and proposition 3.2, we can get

$$\int_H f(h)\ dh = \int_{N_H\backslash Q_H}\int_{\pi\in \mathrm{Irr}_{temp}(G')} \mathcal{B}^{\delta_e,\delta_e}_{\mathcal{D}_{\psi'_N(\pi)}}(T(R(q)f))\ d\mu_{pl}(\pi)\ dq$$

we can show that this double integral converges. Interchanging the order of integration, we get

$$\int_H f(h)\ dh = \int_{\pi\in \mathrm{Irr}_{temp}(G')}\int_{N_H\backslash Q_H} \mathcal{B}^{\delta_e,\delta_e}_{\mathcal{D}^{-1}_{\psi_N}(\pi)}(T(R(q))f)\ dq\ d\mu_{pl}(\pi)$$

By theorem 2.7 and lemma 2.6 we see this is equal to

$$\begin{aligned}
&\int_{\pi\in \mathrm{Irr}_{temp}(G')}\int_{N_H\backslash Q_H} \mathcal{B}^{\ell_H,\delta_e}_{\mathcal{D}_{\psi_N^{-1}}(\mathfrak{S}(\pi))}(T(R(q)f))\ d\mu_{pl}(\pi)\ dq\\
&= \int_{\pi\in \mathrm{Irr}_{temp}(G')}\int_{N_H\backslash Q_H} \mathcal{B}^{\ell_H,\delta_q}_{\mathcal{D}_{\psi_N^{-1}}(\mathfrak{S}(\pi))}(f))\ d\mu_{pl}(\pi)\ dq\\
&= \int_{\pi\in \mathrm{Irr}_{temp}(G')} \mathcal{B}^{\ell_H,\ell_H}_{\mathcal{D}_{\psi_N^{-1}}(\mathfrak{S}(\pi))}(f)\ d\mu_{pl}(\pi)\\
&= \int_{\pi\in \mathrm{Irr}_{temp}(G')} \mathcal{B}^{\ell_H,\ell_H}_{\mathcal{D}_{\psi_N}(\mathfrak{S}(\pi))}(f^\vee)\ d\mu_{pl}(\pi)
\end{aligned}$$

this is equivalent to theorem 1.1.

References

[LO22] Erez Lapid and Omer Offen. Explicit plancherel formula for the space of symplectic forms. *International mathematics research notices*, 2022(18):14255–14294, 2022.