AN EULER SYSTEM OF HEEGNER TYPE

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1. Introduction

This is a study note for Cornut's paper [Cor18], he constructed the Kolyvagin system associated to the special cycles for the spherical pair (SO_{2n+1}, U_n) by reducing the horizontal norm relation to a question on the integral Hecke module structure, I will present the Heegner point case as an example.

Note that there are some extra properties of the spherical pair (SO_{2n+1}, U_n) are used for the Euler system construction to lift the Galois action on the special points to special cycles.

2. The cycles

Let's introduce the ambient group G and its subgroup H. Let F be a totally real number field and pick f_0 in $\mathcal{F} = \mathrm{Spec}(F)(\mathbb{C}) = \mathrm{Spec}(F)(\mathbb{R})$.

Fix a positive integer $n > 0$ and let $(\mathcal{V}, \varphi)$ be a quadratic F-vector space of odd dimension $2n+1$, put

$$(\mathcal{V}_f, \varphi_f) = (\mathcal{V}, \varphi) \otimes_{F,f} \mathbb{R}$$

and suppose that

$$\mathrm{sign}(\mathcal{V}_f, \varphi_f) = \begin{cases} (2n-1, 2) \text{ if } f = f_0 \\ (2n+1, 0) \text{ if } f \neq f_0 \end{cases}$$

set $G = \mathrm{Res}_{F/\mathbb{Q}} SO(\mathcal{V}, \varphi)$. Then we have $G_{\mathbb{R}} = G^{\circ} \times G_{\circ}$ with $G_{\circ} = G_{f_0}$ and $G^{\circ} = \prod_{f \neq f_0} G_f$.

Put $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} (\mathbb{G}_{m,\mathbb{C}})$, we denote $\mathcal{X}$ the $G(\mathbb{R})$-conjugacy class of morphisms $h : S \to G_{\mathbb{R}}$ satisfying the axioms for Shimura varieties, it can be shown that $\mathcal{X}$ can be identified with the space of oriented $\mathbb{R}$-planes in $(\mathcal{V}_0, \varphi_0)$.

Now let's introduce the subgroup, let E be a totally imaginary quadratic extension of F which splits $(\mathcal{V}, \varphi)$, i.e. such that the quadratic space $(\mathcal{V}, \varphi) \otimes_F E$ over E contains a totally isotropic E-subspace of dimension n. We fix once and for all such an E-hermitian F-hyperplane $(\mathcal{W}, \psi) \subset (\mathcal{V}, \varphi)$ and set

$$H = \mathrm{Res}_{F/\mathbb{Q}}\ U(\mathcal{W}, \varphi) \quad G = \mathrm{Res}_{F/\mathbb{Q}}\ SO(\mathcal{V}, \varphi)$$

we define $T = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{m,E}$, $Z = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$, let $T^1 \subset T$ be the kernel of the norm map $N : T \longrightarrow Z$, and denote by $H^1 \subset H$ be the kernel of $\det : \ H \longrightarrow T^1$, so $H^1 = \mathrm{Res}_{F/\mathbb{Q}}\ SU(\mathcal{W}, \psi)$.

We denote

$$\mathcal{X}(H) = \{\ x \in \mathcal{X}\ :\ h_x : S \to G_{\mathbb{R}} \text{ factors through } H_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}\}$$

and we have a decomposition

$$\chi(H) = \mathcal{Y}^+ \sqcup \mathcal{Y}^-$$

for $\mathcal{Y}^{\pm}$ two connected components of $\chi(H)$. It can be shown that $\mathcal{Y} = \mathcal{X}(H)$ can be identified with the space of all negative E_0-lines in $(\mathcal{W}_0, \psi_0)$.

It can be shown that the reflex fields of $\mathcal{X}$ and $\mathcal{Y}^{\pm}$ are $f_0(F)$ and $f_0(E)$, here $f_0(E)$ is a quadratic extension of $f_0(F)$.

For a neat compact open subgroup K of $G(\mathbb{A}_f)$, we denote $Sh_K(G, \mathcal{X})$ the corresponding Shimura variety, it is a quasi-projective smooth algebraic variety over the reflex field $E(G, \mathcal{X}) = F$ with complex points

$$Sh_K(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K \times \mathcal{X}$$

For $g \in G(\mathbb{A}_f)$ and neat open compact subgroup K_1 and K_2 of $G(\mathbb{A}_f)$ such that $g^{-1} K_1 g \subset K_2$, there is a finite etale cover

$$[\cdot g] :\ \mathrm{Sh}_{K_1}(G, \mathcal{X}) \longrightarrow \mathrm{Sh}_{K_2}(G, \mathcal{X})$$

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For the subgroup H, and neat open compact subgroup K' of $H(\mathbb{A}_f)$, we have a projective system of smooth algebraic varieties over the reflex field $E(H, \mathcal{Y}^{\pm}) = E$ with

$$\mathrm{Sh}_{K'}(H, \mathcal{Y}^{\pm})(\mathbb{C}) = H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K' \times \mathcal{Y}^{\pm}$$

for $K' \subset K$, the map given by $H(\mathbb{Q}) \cdot (gK', y) \mapsto G(\mathbb{Q}) \cdot (gK, y)$ induces a finite morphism

$$\mathrm{Sh}_{K'}(H, \mathcal{Y}^{\pm}) \longrightarrow \mathrm{Sh}_K(G, \mathcal{X}) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(E)$$

Let F^{ab} and E^{ab} be the maximal abelian extensions of F and E in $\overline{\mathbb{Q}}$, let $E[\infty]$ be the subfield of E^{ab} which is fixed by the image of the transfer map. Class field theory gives us a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\mathbb{A}_f) & \longrightarrow & T(\mathbb{A}_f) & \longrightarrow & T^1(\mathbb{A}_f) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{Art}_F} & & \downarrow{\scriptstyle \mathrm{Art}_E} & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Gal}(F^{ab}/E) & \longrightarrow & \mathrm{Gal}(E^{ab}/E) & \longrightarrow & \mathrm{Gal}(E[\infty]/E) & \longrightarrow & 1 \end{array}$$

the Art_F and Art_E are the Artin reciprocity maps and the last vertical map induces an isomorphism

$$\mathrm{Art}_E^1 : \ T^1(\mathbb{A}_f)/T^1(\mathbb{Q}) \cong \mathrm{Gal}(E[\infty]/E)$$

Let $\mathcal{B} = (w_1, \cdots w_n)$ be an orthogonal E-basis of $(\mathcal{W}, \psi)$ then

$$T(\mathcal{B}) = \mathrm{Res}_{F/\mathbb{Q}}(U(Ew_1) \times \cdots \times U(Ew_n)) \subset H \subset G$$

this is a maximal $\mathbb{Q}$-subtorus of both H and G with $T(\mathcal{B}) \cong (T^1)^n$.

Let $\mathcal{B}_0 = (w_{1,0}, \cdots, w_{n,0})$ be the orthogonal E_0-basis of (W_0, ψ_0) obtained from $\mathcal{B}$ by base change along $f_0 : F \hookrightarrow \mathbb{R}$, since the signature of $(\mathcal{W}_0, \psi_0)$ equals $(n-1, 1)$, there exists a unqiue i in $\{1, \cdots, n\}$ with $\psi_0(w_{i,0}, w_{i,0}) < 0$, $w_{i,0}$ spans a line $y_{\mathcal{B}} \in (\mathcal{W}_0, \psi_0)$, giving rise to special points $y_{\mathcal{B}}^{\pm} \in \mathcal{Y}^{\pm}$, the corresponding morphisms $h_{\mathcal{B}}^{\pm} : S \to H_{\mathbb{R}}$ or $G_{\mathbb{R}}$ factor through $T(\mathcal{B})_{\mathbb{R}} \hookrightarrow H_{\mathbb{R}}$, we denote the induced cocharacter by $\mu_{\mathcal{B}}^{\pm} : \ G_{m,\mathbb{C}} \to T(\mathcal{B})_{\mathbb{C}}$. By definition, the reflex field $E(G, \mathcal{X})$ is the field of definition of the $G(\mathbb{C})$ conjugacy class of $\mu_{\mathcal{B}}^{+}$.

We define the reflex norm $r_{\mathcal{B}}^{\pm} : f_0 T \to T(\mathcal{B})$ as the composition

$$f_0 T = \mathrm{Res}_{f_0(E)/\mathbb{Q}}(G_{m, f_0(E)}) \longrightarrow \mathrm{Res}_{f_0(E)/\mathbb{Q}}(T(\mathcal{B})_{f_0(E)}) \longrightarrow T(\mathcal{B})$$

where the first map is induced by $\mu_{\mathcal{B}}^{\pm}$ and the second one is the norm.

Now we can describe the reciprocity law for the special points: for every $g \in G(\mathbb{A}_f)$ and $\mathcal{B}$ as above, for any $\sigma \in \mathrm{Aut}(\mathbb{C}, E)$ and $\lambda \in T(\mathbb{A}_f)$ such that $\sigma | E^{ab} = \mathrm{Art}_E(\lambda)$, we have

$$\sigma \cdot [gK, y_{\mathcal{B}}^{\pm}] = [r_{\mathcal{B}}^{\pm}(\lambda) gK, y_{\mathcal{B}}^{\pm}]$$

in $Sh_K(G, \mathcal{X})(\mathbb{C})$. And for every $\lambda \in T^1(\mathbb{A}_f)$

$$\mathrm{Art}_E^1(\lambda) \cdot [gK, y_{\mathcal{B}}^{\pm}] = [\iota_{\mathcal{B}}^{\pm}(\lambda) gK, y_{\mathcal{B}}^{\pm}]$$

where the morphism $\iota_{\mathcal{B}}^{\pm} : T^1 \hookrightarrow T(\mathcal{B}) \subset H \subset G$ is given by

$$\lambda \in T^1 \mapsto (1, \cdots, 1, \lambda^{\pm 1}, 1, \cdots 1) \in T(\mathcal{B}) \cong (T^1)^n$$

here $\lambda^{\pm 1}$ is at the i-th place.

We also have the reciprocity law for connected components: since H^1 is simply connected, and $H^1(\mathbb{Q})$ is dense in $H^1(\mathbb{A}_f)$, it follows that

$$\pi_0(Sh_{K'}(H, \mathcal{Y}^{\pm})) = H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K' = H(\mathbb{Q})H^1(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K'$$

using the determinant map $det : \ H \to T^1$, we get

$$\pi_0(Sh_{K'}(H, \mathcal{Y}^{\pm})) \cong T^1(\mathbb{A}_f)/T^1(\mathbb{Q})\det(K')$$

and for all $\sigma \in Aut(\mathbb{C}/E)$ and $\lambda \in T^1(\mathbb{A}_f)$ such that $Art_E^1(\lambda) = \sigma | E[\infty]$

$$\sigma \cdot C = \lambda^{\pm 1} C$$

for all $C \in \pi_0(Sh_{K'}(H, \mathcal{Y}^{\pm}))$.

For $g \in G(\mathbb{A}_f)$, we will denote $\mathcal{Z}_K(g)$ the image of $gK \times \mathcal{Y}^+$ in

$$\mathrm{Sh}(G,\mathcal{X})(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K \times \mathcal{X}$$

we may view each $\mathcal{Z}_K(g)$ as an subvariety of $\mathrm{Sh}_K(G,\mathcal{X})_{E[\infty]}$, let's define

$$\mathcal{Z}_K(H) = \{\mathcal{Z}_K(g) : \ g \in G(\mathbb{A}_f)\}$$

Proposition 2.1. *The map $g \mapsto \mathcal{Z}_K(g)$ induces a bijection*

$$\mathcal{Z}_K(\bullet) : \ H(\mathbb{Q})\backslash G(\mathbb{A}_f)/K \cong \mathcal{Z}_K(H)$$

3. The Galois action

In this section, I will describe the Galois action on the special cycles. First let's note if $g \in G$ commutes with $T^1 = Z(H)$, then we have $g \in H$ and hence $H = Z_G(T^1)$, it follows that T^1 and H have the same normalizer N in G.

Since K is open in $G(\mathbb{A}_f)$, we also have

$$\mathcal{Z}_K(\cdot) : \ \overline{H(\mathbb{Q})}\backslash G(\mathbb{A}_f)/K \cong \mathcal{Z}_K(H)$$

where $\overline{H(\mathbb{Q})}$ is the closure of $H(\mathbb{Q})$ in $G(\mathbb{A}_f)$. Since the derived subgroup H^1 of H is simply connected, $H^1(\mathbb{Q})$ is dense in $H^1(\mathbb{A}_f)$ by strong approximation, since E is a CM extension of F, $T^1(\mathbb{Q})$ is discrete and thus closed in $T^1(\mathbb{A}_f)$. It follows that

$$H(\mathbb{Q}) \cdot H^1(\mathbb{A}_f) \subset \overline{H(\mathbb{Q})} \subset \det{}^{-1}(T^1(\mathbb{Q})) = H(\mathbb{Q}) \cdot H^1(\mathbb{A}_f)$$

i.e. $\overline{H(\mathbb{Q})} = H(\mathbb{Q})H^1(\mathbb{A}_f)$ in $H(\mathbb{A}_f)$.

$\overline{H(\mathbb{Q})} = H(\mathbb{Q})H^1(\mathbb{A}_f)$ is a normal subgroup of $N(\mathbb{Q})H(\mathbb{A}_f)$ and the quotient group acts on $\mathcal{Z}_K(H)$ by left multiplication in $G(\mathbb{A}_f)$, the quotient group is a generalized dihedral extension

$$1 \to \frac{T^1(\mathbb{A}_f)}{T^1(\mathbb{Q})} \cong \frac{H(\mathbb{A}_f)}{H(\mathbb{Q})H^1(\mathbb{A}_f)} \to \frac{N(\mathbb{Q})H(\mathbb{A}_f)}{H(\mathbb{Q})H^1(\mathbb{A}_f)} \to \frac{N(\mathbb{Q})}{H(\mathbb{Q})} \cong \{\pm 1\}$$

with a natural splitting comes from $N(\mathbb{Q}) \hookrightarrow N(\mathbb{Q})H(\mathbb{A}_f)$, giving rise to

$$\det : \ N(\mathbb{Q})H(\mathbb{A}_f)/H(\mathbb{Q})H^1(\mathbb{A}_f) \cong T^1(\mathbb{A}_f)/T^1(\mathbb{Q}) \rtimes \{\pm 1\}$$

which is isomorphic to the dihedral Galois extension

$$1 \longrightarrow \mathrm{Gal}(E[\infty]/E) \longrightarrow \mathrm{Gal}(E[\infty]/F) \longrightarrow \mathrm{Gal}(E/F) \longrightarrow 1$$

endowed with the canonical splitting given by the complex conjugation

$$\mathrm{Gal}(E[\infty]/F) \cong \mathrm{Gal}(E[\infty]/E) \rtimes \{1, c\}$$

we will denote the resulting extension of $\mathrm{Art}^1_E : \ T^1(\mathbb{A}_f)/T^1(\mathbb{Q}) \cong \mathrm{Gal}(E[\infty]/E)$ by

$$\mathrm{Art}^1_{E/F} : \ T^1(\mathbb{A}_f)/T^1(\mathbb{Q}) \rtimes \{\pm 1\} \cong \mathrm{Gal}(E[\infty]/F)$$

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Proposition 3.1. *For all $g \in G(\mathbb{A}_f)$, $\sigma \in Aut(\mathbb{C}/F)$ and $\lambda \in N(\mathbb{Q})H(\mathbb{A}_f)$ such that $Art^1_{E/F} \circ det(\lambda) = \sigma| \ E[\infty]$, we have $\sigma \cdot \mathcal{Z}_K(g) = \mathcal{Z}_K(\lambda g)$.*

For $\sigma \in Aut(\mathbb{C}/E)$, this is the Galois action on connected components.

4. The Kolyvagin system of Heegner point

Kolyvagin studied a collection of distinguished points on modular curves $X_0(N)$ known as the Heegner points, the starting point of Kolyvagin's argument is the well-known trace relation

$$\mathrm{Tr}_\ell(x_{n\ell}) = T_\ell x_n$$

for $\ell \nmid n$, here $x_m \in X_0(N)$ denotes the Heegner point defined over the ring class extension $E[m]$ of conductor m of an imaginary quadratic field E, ℓ is a rational prime that is inert in E and T_ℓ is the self-dual Hecke correspondence of bidegree $\ell+1$ and Tr_ℓ the trace map from $E[n\ell]$ to $E[n]$. The operator T_ℓ essentially computes the local L-factor at the prime ℓ of the modular curve parametrized by $X_0(N)$, T_ℓ is essentially the middle coefficient of a degree two Hecke polynomial, the ideal version of the trace relation should involve the complete Hecke polynomial.

Let E be an imaginary quadratic field, for $G = GL_2/\mathbb{Q}$, $H = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$, fix an isomorphism $E \cong \mathbb{Q}^2$ of $\mathbb{Q}$-vector spaces, this will induce an inclusion of algebraic groups

$$\iota :\ H \hookrightarrow G$$

let $(G, \mathcal{X}_{std})$ be the Shimura data associated with G, note $(H, \{h_0\})$ forms a Shimura data, where $h_0 :\ S \cong H_{\mathbb{R}}$, hence we have a morphism of Shimura data

$$\iota :\ (H, \{h_0\}) \to (G, \mathcal{X}_{std})$$

We will denote K a fixed open compact subgroup of $G(\mathbb{A}_f)$, fix S any subset of rational primes ℓ of $\mathbb{Q}$ such that

- ℓ is unramified at E.
- K is hyperspecial at ℓ.
- H is unramified at ℓ.
- If ℓ is inert, K^ℓ contains $\mathrm{diag}(\ell,\ell) \hookrightarrow G(\mathbb{A}_f^\ell)$.

Let $\mathcal{N}$ be the set of all square-free products of primes in S, for $n \in \mathcal{N}$, we will write $K = K^{[n]}K_{[n]}$ where $K_{[n]} := \prod_{\ell|n} K_\ell$ and $K^{[n]} \subset G(\mathbb{A}_f^{[n]})$.

Definition 4.1. The group of CM divisors on $\mathcal{S}_K$ is defined to be the free $\mathbb{Z}$-module $\mathcal{Z} = \mathcal{Z}_K := \mathbb{Z}[\mathcal{T}_K(\mathbb{C})]$.

Let's denote

$$\sigma_\ell := \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \quad \tau_\ell := \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}$$

Definition 4.2. The Hecke polynomial at a prime $\ell \in S$ is the polynomial

$$H_\ell(X) = \ell \cdot 1_K - 1_{K\sigma_\ell K}X + 1_{K\tau_\ell K}X^2$$

Remark 4.3. The Hecke polynomial naturally arises from the cocharacter associated with the Shimura data.

The group of CM divisors has a left action of the Galois group, which is the same as the action of $H(\mathbb{A}_f)$, it is given by

$$h_f[x_\iota, g_f]_K = [x_\iota, h_f g_f]_K$$

where $h_f \in H(\mathbb{A}_f)$, $g_f \in G(\mathbb{A}_f)$.

Let's denote $\mathcal{F}$ the quotient of $C_c(G(\mathbb{A}_f)/K)$ by submodule generated by the relation $\xi - \xi(h^{-1}(-))$ for $h \in H(\mathbb{Q}) = E^\times$ and $\xi \in C_c(G(\mathbb{A}_f)/K)$. The module $\mathcal{F}$ inherts the action of $H(\mathbb{A}_f)$ and of the Hecke operatrs $ch(KgK),\ g \in G(\mathbb{A}_f)$, there is an isomorphism of abelian groups

$$\psi :\ \mathcal{F} \to \mathcal{Z}$$
$$[ch(g_1K)] \mapsto [x_\iota, g_1]_K$$

For $\ell \in S$, we define $g_\ell \in GL_2(\mathbb{Q}_\ell)$, and subgroups $H_{g_\ell} := H(\mathbb{Q}_\ell) \cap g_\ell K_\ell g_\ell^{-1}$. we set

$$H_n = (H(\mathbb{A}_f^{[n]}) \cap K^{[n]}) \cdot \prod_{\ell|n} H_{g_\ell}$$

For $n \in \mathcal{N}$, let $E[n]$ be the abelian field extension of E corresponding to H_n, i.e., $E[n]$ is the field such that $\mathrm{Gal}(E^{ab}/E[n])$ is identified with $E^\times\backslash E^\times H_n \subset H(\mathbb{Q})\backslash H(\mathbb{A}_f)$.

Let $Tr_{E[n\ell]/E[n]} : \mathcal{Z}(H_{n\ell}) \to \mathcal{Z}(H_n)$, the trace map induced by summing over elements in $\mathrm{Gal}(E[n\ell]/E[n])$.

Theorem 4.4. *For $n \in \mathcal{N}$, there exists n-th Euler system divisor class $y_n \in \mathcal{Z}(H_n)$, such that for $\ell \in S$ with $\ell \nmid n$, we have*

$$H_\ell(Frob_\lambda) y_n = Tr_{E[n\ell]/E[n]}(y_{n\ell})$$

The proof of this theorem is reduced to show that there exists *text vectors* $\zeta_n \in C_c(G(\mathbb{Q}_{[n]})/K_{[n]})$ such that

$$H_\ell(h_\ell) \cdot [\zeta_n] = \sum_{\gamma \in H_n/H_{n\ell}} \gamma \cdot [\zeta_{n\ell}]$$

5. Norm relation

We state the norm relation in general in this section. The local horizontal norm relation follows from the following local result: for $\mathfrak{p} \notin S$, we have the Satake isomorphism

$$\mathcal{H}_\mathfrak{p} = \mathbb{Z}[c_{\mathfrak{p},1}, \cdots, \ c_{\mathfrak{p},n}]$$

we will denote $c_{\mathfrak{p},n}$ by $T_\mathfrak{p}$.

Theorem 5.1. *For every $\ell \in \mathcal{P}$, there is an element*

$$\circ_\ell^b \in S(H^1(F_\ell)\backslash G(F_\ell)/G(\mathcal{O}_{F,\ell}))^{U_\ell^1(1)}$$

such that

$$Tr_\ell(\circ_\ell^b) = T_\ell \cdot \circ_\ell$$

in $S(H^1(F_\ell)\backslash G(F_\ell)/G(\mathcal{O}_{F,\ell}))^{U_\ell^1(0)}$, here Tr_ℓ is the usual trace operator over $U_\ell^1(0)/U_\ell^1(1) \cong \mathbb{G}(\ell)$.

References

[Cor18] Christophe Cornut. An Euler system of Heegner type. *preprint*, 22, 2018.