# EQUIVARIANT COHOMOLOGY

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## 1. Introduction

This is a study note for equivariant cohomology based on Brion's paper [Bri98] and the book by Bernstein-Lunts, Brion's paper is on the topological setting and Bernstein-Lunts' book works with the equivariant derived category.

## 2. Topological setting

2.1. **Equivariant cohomology.** We want to find explicit description of cohomology ring of certain manifolds with group actions which arise in representation theory, e.g. homogeneous spaces, compact multiplicity free spaces.

In this section, we will assume that $k = \mathbb{Q}$, $X$ a topological space, $H^*(X) = H^*(X, \mathbb{Q})$, $H^*(X)$ is a graded $\mathbb{Q}$-algebra. $f : X \to Y$ continuous map between topological spaces will induce $f^* : H^*(Y, k) \to H^*(X, k)$. Given $X$ with a topological group $G$-action, we will say $X$ is a $G$-space.

There exists a principal $G$-bundle

$$p : E_G \longrightarrow B_G$$

$E_G$ is contractible, such a bundle is universal among principal $G$-bundles. $G$ acts on $X \times E_G$ diagonally and the quotient $X \times_G E_G := X \times E_G/G$ exists, we define

$$H^*_G(X, k) := H^*(X \times_G E_G, k)$$

In particular, if $X$ is a point, we get

$$H^*_G(pt) = H^*(E_G/G) = H^*(B_G, k)$$

the projection $p_x : X \times E_G \to E_G/G$ is a projection with fiber $X$. $H^*_G(X)$ is an algebra over $H^*_G(pt)$.

Let's consider some special cases:

i) If $G$ acts trivially on $X$, then $X \times_G E_G = X \times B_G$, by Kunneth isomorphism we have

$$H^*_G(X) = H^*(X) \times H^*_G(pt)$$

ii) If $G$ acts on $X$ and $X/G$ exists, then

$$\pi : X \times_G E_G \longrightarrow X/G$$

is a map with fiber $E_G/G_x$ at $x \in X/G$, for $G_x$ the isotropy group of $x \in X$, $G_x$ is finite and $E_G$ is contractible, we have $E_G/G_x$ is $\mathbb{Q}$-acyclic, and $\pi^* : H^*(X/G) \to H^*_G(X)$ is an isomorphism.

iii) Let $H$ be a closed subgroup of $G$, then $E_G/H$ exists and $E_G \to E_G/H$ is a universal bundle for $H$, hence

$$H^*_G(G/H) = H^*(G/H \times_G E_G) = H^*_H(pt, k)$$

more generally, for $Y$ a $H$-space, we have

$$H^*_G(G \times_H Y, k) \cong H^*_H(Y, k)$$

Restriction to the fiber of $p_X$ defines an algebra homomorphism

$$\rho : H^*_G(X)/(H^+_G(pt)) = H^*_G(X) \otimes_{H^*_G(pt)} k \longrightarrow H^*(X)$$

here $(H^+_G(pt))$ is the ideal of $H^*_G(X)$ generated by homogeneous elements of $H^*_G(pt)$ of positive degree.

Our main result is that, for certain spaces $X$ and for rational coefficients the map $\rho$ is an isomorphism and the equivariant cohomology algebra can be described completely 2.6, 2.10.

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**Example 2.1.** Let $X$ be a space with $G = S^1$-action, the action $S^1 \times X \to X$ makes $H^*(X)$ into an $\Lambda = k[y]/y^2$ module with $y$ of cohomological degree -1. Denote $S = H^*(BS^1)$, we have the following relation between the equivariant cohomology and ordinary cohomology

$$H^*(X) = H^*_{S^1}(X) \otimes_S k$$
$$H^*_{S^1}(X) \cong \mathrm{Hom}_\Lambda(k, H^*(X))$$

here we note $X = pt \times_{pt/G} X//S^1$ amd $X//S^1 = X^{S^1}$, where $X//S^1 = ES^1 \times_{S^1} X$. *The equivariant cohomology for a $S^1$-space $X$ and ordinary cohomology of $X$ are Koszul dual to each other.*

**Proposition 2.2.** *Let $G$ be a compact connected Lie group and let $T \subset G$ be a maximal torus with normalizer $N$ and with Weyl group $W = N/T$, let $X$ be a $G$-space. Then*

- *The group $W$ acts on $H^*_T(X)$ and we have an isomorphism*
$$H^*_G(X) \cong H^*_T(X)^W$$
*in particular, we have $H^*_G(pt)$ is isomorphic to $S^W$ for $S$ the symmetric algebra of $\chi(T)$ and $S^W$ the ring of $W$-invariants in $S$.*
- *The map*
$$S \cong H^*_G(G/T) \to H^*(G/T)$$
*is surjective and induces isomorphism $S/(S^W_+) \to H^*(G/T)$ where $(S^W_+)$ denotes the ideal of $S$ generated by all homogeneous $W$-invariants of positive degree.*
- *We have an isomorphism*
$$S \otimes_{S^W} H^*_G(X) \cong H^*_T(X)$$
*in particular, we have $H^*_T(G/T)$ is isomorphic to $S \otimes_{S^W} S$.*

*Proof.* For the first statement, we denote $G^{\mathbb{C}}$ the complexification of $G$, let $B$ be a Borel subgroup of $G^{\mathbb{C}}$ containing the compact torus $T$, by the Iwasawa decomposition, we have $G^{\mathbb{C}} = GB$, and $G \cap B = T$, the map $G/T \to G^{\mathbb{C}}/B$ is a homeomorphism, the flag manifold has a stratification by $|W|$ strata, each isomorphic to a complex affine space, it follows $H^*(G/T)$ vanishes in odd degrees. The Weyl group acts freely on $G/T$ with quotient $G/N$ we have an isomorphism

$$H^*(G/N) \cong H^*(G/T)^W$$

moreover $\chi(G/N) = |W|^{-1}\chi(G/T) = 1$, we have $H^*(G/N)$ vanishes in odd degrees and is one-dimensional it follows that $G/N$ is $\mathbb{Q}$-acyclic. The fibration

$$X \times_N E_G \to X \times_G E_G$$

with fiber $G/N$ induces an isomorphism $H^*_G(X) \to H^*_N(X)$, we have a covering

$$X \times_T E_G \to X \times_N E_G$$

with fiber $W$, hence $H^*_G(X) = H^*_N(X) \cong H^*_T(X)^W$.

For the second statement, applying the previous result to a point, we get

$$H^*(B_G) = H^*_G(pt) = S^W$$

the odd degree part is zero. The fibration

$$G/T \times_G E_G \to B_G$$

has fiber $G/T$ with vanishing odd cohomology. The Leray spectral sequence degenerates and we obtain an isomorphism

$$H^*(G/T) \cong H^*_G(G/T)/(H^*(B_G)_+) = H^*_T(pt)/(H^*_T(pt)^W_+) = S/(S^W_+)$$

For the third statement, from the fibration

$$X \times_T E_G \longrightarrow X \times_G E_G$$

with fiber $G/T$, restriction to a fiber gives a ring homomorphism $H^*_T(X) \to H^*(G/T)$ which is surjective, the Leray spectral sequence degenerates, and

$$H^*_T(X) \cong H^*_G(X) \otimes H^*(G/T)$$

$\square$

For $G$ and $X$ as above, consider the restriction map

$$\rho : H^*_G(X)/(S^W_+) \to H^*(X)$$

$\rho$ may not be surjective, there is an important class of $G$-spaces for which $\rho$ is an isomorphism.

**Proposition 2.3.** *Let $G$ be a connected compact Lie group and let $X$ be a compact Hamiltonian $G$-space, the notaion as above, then the $S^W$-module $H^*_G(X)$ is free and the map $\rho : H^*_G(X)/(S^W_+) \to H^*(X)$ is an isomorphism.*

As an example, we may consider the flag manifold, $X = G/T$, as a coadjoint orbit $G/T$ is a Hamiltonian $G$-manifold and the moment map is an inclusion, we recover the description of the cohomology ring of $X$ $H^*(X) \cong S/(S^W_+)$.

2.2. **Localization theorem.** A powerful tool in the study of equivariant cohomology is the following localization theorem due to Borel-Atiyah-Segal

**Theorem 2.4.** *Let $T$ be a compact torus and let $X$ be a $T$-space which embeds equivariantly into a finite dimensional $T$-module, then let $i_T : X^T \to X$ be the inclusion of the fixed point set, then the $S$-linear map*

$$i^*_T : H^*_T(X) \longrightarrow H^*_T(X^T)$$

*becomes an isomorphism after inverting finitely many characters of $T$.*

The proof uses the following lemma

**Lemma 2.5.** *Let $G$ be a connected compact Lie group and let $\Gamma \subset G$ be a closed subgroup with centralizer $G^\Gamma$ and $X$ a symplectic $G$-manifold, then the fixed point set $X^\Gamma$ is a symplectic $G^\Gamma$ manifold and the normal bundle $N_{X,X^\Gamma}$ has a natural structure of a complex vector bundle.*

*If moreover the $G$-action on $X$ is Hamiltonian with moment map $\mu$, then the $G^\Gamma$-action on $X^\Gamma$ is Hamiltonian with moment map: restriction of $\mu$ followed by restriction to $\mathfrak{g}^\Gamma$.*

As before, we consider a compact torus $T$ and a $T$-space, we denote by $i_T : X^T \to X$ the inclusion of the fixed point set. Under the condition that $X$ is a $T$-space which admits an equivariant embedding into the space of a finite-dimensional representation of $T$, if the $S$-module $H^*_T(X)$ is free, then the map

$$i^*_T : H^*_T(X) \longrightarrow H^*_T(X^T)$$

is injective,, and its image is the restriction of the images of the maps

$$i^*_{T,T'} : \ H^*_T(X^{T'}) \longrightarrow H^*_T(X^T)$$

where $T'$ runs over all subtori of codimension 1 of $T$.

When $X$ is Hamiltoinan, we can obtain the following more precise version of the localization theorem

**Theorem 2.6.** *Let $X$ be a compact Hamiltonian $T$-spaces with finitely many fixed points $x_1 \cdots x_m$ and $dim(X^{\Gamma'}) \leq 2$ for any subtori $T' \subset T$ of codimension 1. Then via the map $i^*_T$, the algebra $H^*_T(X)$ is isomorphic to the subalgebra $S^m$ consisting of all $m$-tuples $(f_1, \cdots, f_m)$ such that: $f_j \equiv f_k \pmod{\chi}$, where the fixed points $x_j$ and $x_k$ are in the same connected component of $X^{ker(\chi)}$ for $\chi$ a primitive character of $T$. Moreover, the cohomology algebra $H^*(X)$ is the quotient of $H^*_T(X)$ by the ideal generated by $(f, f, \cdots, f)$ where $f \in S$ is homogeneous of positive degree.*

**Example 2.7.** Let $G$ be a connected compact Lie group and $T \subset G$ a maximal torus, and $X = G/T$ the flag manifold of the complexification of $G$. Denote $x$ the base point of $X$, then the fixed point $X^T$ is the orbit $Wx$, denote by $\Phi$ the root system of $(G,T)$. Let $\chi$ be a primitive character of $T$, if $\chi$ is not in $\Phi$, then $X^{\ker(\chi)} = X^T$. If $\chi$ is in $\Phi$, then

$$X^{\ker(\chi)} = G^{\ker(\chi)} Wx$$

is a disjoint copy of $|W|/2$ complex projective lines joining the fixed points $\omega x$ and $s\omega x$ for all $\omega \in W$, so we obtain from the theorem 2.6

$$H^*_T(G/T) = \{(f_\omega)_{\omega \in W} \mid f_\omega \in S, \ f_\omega \equiv f_{s_\alpha \omega} \pmod{\alpha} \ \forall \alpha \in \Phi, \ \forall \omega \in W\}$$

*Remark* 2.8. We note that the calculation of the equivariant cohomology of flag manifolds plays an important role in the theory of Koszul duality for category $\mathcal{O}$ [Soe98].

2.3. **Equivariant cohomology of multiplicity free manifolds.** We want to generalize the description of the equivariant cohomology in theorem 2.6 to the class of all compact multiplicity free Hamiltonian spaces.

**Definition 2.9.** A Hamiltonian $G$ $X$ is *multiplicity-free* if $X$ is connected and the preimage under the moment map of any coadjoint $G$-orbit consists of finitely many $G$-orbits.

If moreover $X$ is compact, then the fibers of the moment map are connected and the multiplicity-free amounts to: the preimage of each orbit under the moment map is an unique orbit.

**Theorem 2.10.** *Let $X$ be a compact multiplicity-free space under a connected compact Lie group $G$, then with the notation above, the algebra $H_G^*(X)$ is isomorphic via $i_T^*$ to the algebra $S^m$ consisting of all $m$-tuples $(f_1, \cdots, f_m)$ such that*

- *each $f_j$ is in $S^{W_j}$.*
- *$f_j \equiv \omega(f_k)$ (mod $\lambda_j - \omega(\lambda_k)$) whenever $\omega \in W$ and the segment $[\lambda_j, \omega(\lambda_k)]$ is a component of $\mu(X_1) \cap \mathfrak{t}^*$, here $X_1$ is the set of all $x \in X$ such that the rank of the isotropy group $G_x$ is at least $rk(G) - 1$.*

*moreover $i_T^*(S^W)$ consists of all tuples $(f, \cdots, f)$ where $f \in S^W$.*

The proof uses a sharper form of the lemma 2.5.

**Example 2.11.** Let $X$ be the $G$-orbit of $\lambda \in \mathfrak{g}^*$, we may assume that $\lambda \in \mathfrak{t}_+^*$, let $\mu : X \to \mathfrak{g}^*$ be the inclusion map, hence $\mu(X^T) \cap \mathfrak{t}^* = \{\lambda\}$ and $\mu(X_1) \cap \mathfrak{t}^* = W \cdot \lambda$, then the previous theorem reduces to $H_G^*(G\lambda) = S^{W_\lambda}$ which follows more directly from

$$H_G^*(G\lambda) = H_G^*(G/G_\lambda) = H_{G_\lambda}^*(pt)$$

## References


[Bri98] Michel Brion. Equivariant cohomology and equivariant intersection theory. In *Representation theories and algebraic geometry*, pages 1–37. Springer, 1998.

[Soe98] Wolfgang Soergel. Combinatorics of harish-chandra modules. In *Representation theories and algebraic geometry*, pages 401–412. Springer, 1998.