ENDOSCOPIC CLASSIFICATION OF REPRESENTATIONS

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1. Introduction

This is a study note on Arthur's endoscopic classification based on the chapter "Endoscopic classification of representations" from the book [GH].

2. Results

In [Art13], Arthur proves the existence of functorial transfer with respect to the L-maps r. In particular, there is no genericity assumption in his work. Moreover, he gave a precise enough description of the fibers of the functorial transfer that he could classify the discrete spectrum of $L^2([G_n])$ in terms of the automorphic representations on H_N. We will explain these results in this section.

The main tool used in Arthur's result is the theory of **twisted endoscopy**, so one refers to Arthur's work and subsequent refinements as the **endoscopic classification** of representations.

In his book [Art13], Arthur gives a careful account of how to replace objects attached to the conjectural global Langlands dual group $\mathcal{L}_F$. We will not use this object and we will state Arthur's main result as directly as possible.

In the reminder of this section, we assume that $G_n \neq U_n$.

Theorem 2.1. *Every irreducible subrepresentation of $L^2([G_n])$ admits a functorial transfer to $H_N(\mathbb{A}_F)$ with respect to r.*

To make precise what one means by a functorial transfer, one needs more than the theory of the local Langlands conjecture as some irreducible subrepresentations of $L^2([G_n])$ need not to be tempered.

Let

$$L^2_{\text{disc}}([G_n]) \subset L^2([G_n])$$

be the largest closed subspace that decomposes discretely under $G_n(\mathbb{A}_F)$, in view of the theorem 2.1, it is natural to partition $L^2_{\text{disc}}([G_n])$ into the fibers of the functorial transfer to $H_N(\mathbb{A}_F)$ and then try to describe the fibers. This is precisely what Arthur accomplished.

Let $\tilde{C}_c^\infty(G_n(\mathbb{A}_F))$ be $C_c^\infty(G_n(\mathbb{A}_F))$ except in the special case where G_n is SO_{2n} or SO^*_{2n}, in which case it is subalgebra of $C_c^\infty(G_n(\mathbb{A}_F))$ invariant under θ, the automorphism induced by conjugation.

We state the main classification result first:

Theorem 2.2. *(Arthur) There is an $\tilde{C}_c^\infty(G_n(\mathbb{A}_F))$-module isomorphism*

$$L^2_{disc}([G_n]) \cong \bigoplus_{\psi \in \tilde{\Psi}_2(G_n)} \bigoplus_{\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)} \pi^{\oplus m_\psi}$$

For a cuspidal autormophic representation τ of $H_N(\mathbb{A}_F)$ and $m \in \mathbb{Z}$, let (τ, m) be the Speh representation, one says that τ is of **orthogonal type** if $L(s, \tau, \text{sym}^2)$ has a pole at $s = 1$ and of **symplectic type** if $L(s, \tau, \wedge^2)$ has a pole at $s = 1$.

Since

$$L(s, \tau, Sym^2) L(s, \tau, \wedge^2) = L(s, \tau \times \tau)$$

we have τ cannot be both orthogonal and symplectic.

The set $\tilde{\Psi}_2(G_n)$ is the set of automorphic representations of $H_N(\mathbb{A}_F)$ of the form

$$\boxtimes_{i=1}^d \ (\tau_i, m_i)$$

where

Date: September 2025.

- τ_i is a cuspidal automorphic representation of $H_{N_i}(\mathbb{A}_F)$.
- $\sum_{i=1}^d N_i m_i = N$.
- $\tau_i^\vee \cong \tau_i$ for all i.
- $\tau_i \cong \tau_j$ if and only if $i = j$.
- If ${}^L G_n^\circ$ is orthogonal (resp. symplectic), then (τ_i, m_i) is orthogonal (resp. symplectic).

The set $\tilde{\Psi}_2(G_n)$ is known as the set of **discrete global A-parameters** of G_n. The discrete global A-parameter is said to be *generic* if $m_i = 1$ for all i. In this case, we also refer to the parameter as a **discrete generic global L-parameter**.

For every $\psi \in \tilde{\Psi}_2(G_n)$, Arthur defines a finite 2-group $\mathcal{S}_\psi$ and a character

$$\epsilon_\psi : \mathcal{S}_\psi \longrightarrow \{\pm 1\}$$

for every place v of F and every $\psi \in \widetilde{\Psi}_2(G_n)$ we define a representation

$$\psi_v : W'_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L H_N$$

by

$$\psi_v = \bigoplus_{i=1}^d \mathrm{rec}(\tau_{iv}) \otimes \mathrm{Sym}^{m_i}$$

the extra $SL_2(\mathbb{C})$ factor occuring in the domain of ψ_v is known as the Arthur-SL_2 and plays a role similar to the representation of SL_2 that appear in the Hodge theory.

One proves that an $\widehat{H}_N(\mathbb{C})$-conjugate of ψ_v factors through the L-map r and hence ψ_v defines a homomorphism

$$\psi_v : W'_{F_v} \times SL_2(\mathbb{C}) \longrightarrow {}^L G_n$$

these are examples of **local A-parameters**.

One shows in addition the existence of maps

$$\mathrm{loc}_v : \mathcal{S}_\psi \longrightarrow \pi_0(\overline{\mathcal{S}}_{\psi_v})$$

where

$$\overline{\mathcal{S}}_{\psi_v} = C_{\hat{G}_n(\mathbb{C})}(\mathrm{im}(\psi_v))/Z^{\mathrm{Gal}_{F_v}}(\hat{G}_n(\mathbb{C}))$$

For each A-parameter, Arthur defines a set $\tilde{\Pi}(\psi_v)$ of irreducible admissible representations of $G_n(F_v)$ satisfying certain desiderata.

Any $\pi_v \in \tilde{\Pi}(\psi_v)$ comes with a character

$$\langle \pi_v, \cdot \rangle : \pi_0(\overline{\mathcal{S}}_{\psi_v}) \longrightarrow \mathbb{C}^\times$$

and thus for all $\pi \in \tilde{\Pi}(\psi)$, one obtain a character

$$\langle \pi, \cdot \rangle = \prod_v \langle \pi_v, \cdot \rangle$$

this allows us to define the global adelic A-packet

$$\tilde{\Pi}(\psi) := \{\otimes'_v \pi_v \ : \ \pi_v \in \tilde{\Pi}(\psi_v) \text{ and } \langle \pi_v, \cdot \rangle = 1 \text{ for almost all v}\}$$

it consists of a set of admissible representations of $G_n(\mathbb{A}_F)$.

The last piece of the classification theorem is determining which occur in $L^2([G_n])$, this is provided by ϵ_ψ. One defines

$$\tilde{\Pi}_\psi(\epsilon_\psi) = \{\pi \in \tilde{\Pi}(\psi) \ : \ \langle \pi, \cdot \rangle = \epsilon\}$$

Local L-packets for $H_{N_{F_v}}$ are singletons for all places v of F, at least in the almost tempered case. Hence global L-parameters into ${}^L H_N$ that are direct sums of discrete generic global L-parameters should correspond bijectively to isobaric sums

$$\boxtimes_{i=1}^k \pi_i$$

of cuspidal autormophic representations of $A_{GL_{n_i}} \backslash GL_{n_i}(\mathbb{A}_F)$ satisfying

$$\sum_{i=1}^k n_i = N$$

this bijection should be compatible with the local Langlands correspondence. On the other hand, a discrete generic global parameter into ${}^L G_n$ is a particular type of homomorphism $\mathcal{L}_F \longrightarrow {}^L H_N$ that factors through the L-map $r:\ {}^L G_n \to {}^L H_N$. Arthur identifies the set of discrete generic global parameters $\mathcal{L}_F \to {}^L H_N$ with the set of isomorphism classes of cuspidal representations of $A_{GL_N} \backslash GL_N(\mathbb{A}_F)$. He then isolates exactly which isobaric sums would have global L-parameters into ${}^L H_N$ that factors through $r:\ {}^L G_n \to {}^L H_N$ if we knew $\mathcal{L}_F$ existed.

If we knew the existence of the global Langlands group $\mathcal{L}_F$, then the discrete generic global L-parameters $\rho:\ \mathcal{L}_F \to {}^L H_{2n+1}$ that factor through

$$r:\ {}^L Sp_{2n} \longrightarrow {}^L GL_{2n+1}$$

would be precisely be those ρ whose image under the projection

$${}^L GL_{2n+1} \longrightarrow GL_{2n+1}(\mathbb{C})$$

fixes a symmetric bilinear form on $\mathbb{C}^{2n+1}$.

By a generalization of the Artin conjecture, the trivial representation occurs in $\mathrm{Sym}^2 \circ \rho$ if and only if $L(s, \mathrm{Sym}^2 \circ \rho)$ has a pole at $s = 1$. The translation to the automorphic side of the conjectural global Langlands correspondence is the assertion that a cuspidal automorphic representation π on $GL_{2n+1}(\mathbb{A}_F)$ is a functorial transfer from $Sp_{2n}(\mathbb{A}_F)$ if and only if $L(s, \pi, \mathrm{Sym}^2)$ has a pole at $s = 1$.

The line of reasoning as above also leads to the expectation that the cuspidal autormophic representations π' of $A_{H_N} \backslash H_N(\mathbb{A}_F)$ that are functorial transfer from $G_n(\mathbb{A}_F)$ with respect to $r:\ {}^L G_n \to {}^L H_N$ are precisely those π' such that $L(s, \pi', r')$ has a pole at $s = 1$

G_n	r'
SO_{2n+1}	$\wedge^2$
SO_{2n}	Sym^2
SO_{2n}^*	Sym^2
Sp_{2n}	Sym^2
U_{2n}	$\mathrm{As}_{E/F} \quad \otimes$
	$\eta_{E/F}$
U_{2n+1}	$\mathrm{As}_{E/F}$

References

[Art13] James Arthur. *The Endoscopic classification of representations orthogonal and symplectic groups*, volume 61. American Mathematical Soc., 2013.

[GH] Jayce R Getz and Heekyoung Hahn. An Introduction to Automorphic Representations with a view toward trace formulae.