PERIODS OF EISENSTEIN SERIES

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1. Introduction

This is a study note on periods of Eisenstein series, we presenting two methods here: The first method is the soft method in the paper [Sak13] of Sakellaridis based on the philosophy of linking periods of automorphic forms with local Plancherel formulas. The second method is the hard method in the [JLR99] paper.

2. Periods and local Plancherel formula

Let F be a number field and G a split connected reductive group defined over the ring of integers of F (as we will see, the split condition is not necessary here once we have done the relevant local unramified computation), assume that $B(F)$ has a single orbit on $\mathring{X}(F)$, we also assume that for almost all completions F_v of F, the variety X_{F_v} satisfies the various assumptions in the paper [Sak13]. We will denote by $A(\mathbb{A}_F)^1$ the intersection of the kernels of all homomorphisms: $A(\mathbb{A}_F) \to \mathbb{G}_m(\mathbb{A}_f) \to \mathbb{R}_+^\times$ where the first denotes an algebraic character and the second denotes the absolute value.

Any idele class character of $A(\mathbb{A}_F)$ can be twisted by characters of the group $A(\mathbb{A}_F)/A(\mathbb{A}_F)^1$, and thus lives in a $rk(A)$-dimensional complex manifold of characters. Let ω be such a family, for $\chi \in \omega$, we denote $I(\chi) = \mathrm{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\chi\delta^{1/2})$, the normalized principal series of $G(\mathbb{A}_F)$, considered as a holomorphic family of vector spaces, that is we fix a notion of "holomorphic sections" by identifying the underlying vector spaces of the representations in the usual way.

For a meromorphic family of sections $\chi \mapsto f_\chi \in I(\chi)$, we have the principal Eisenstein series defined by the convergent sum:

$$E(f_\chi, g) := \sum_{\gamma \in B(F)\backslash G(F)} f_\chi(\gamma g)$$

if $\langle \check{\alpha}, \mathrm{Re}(\chi)\rangle >> 0$ for all $\alpha \in \Delta$ and by meromorphic continuation to the whole ω.

Let H be a spherical group of G over F, we would like to compute the period integral $\int_{H(F)\backslash H(\mathbb{A}_F)} E(f_\chi, h)\, dh$, of course, this integral may not be convergent, therefore we have to understand it as a distribution on the Eisenstein spectrum of G, our goal is to compute the most continuous part of this distribution. We notice that regularized periods of Eisenstein series abound in the literature, and different methods are suitable for different purposes. The method in the paper [Sak13] is much softer than most, and it helps motivate the general philosophy linking periods of automorphic forms with local Plancherel formulas.

For simplicity, we will discuss only the case where ω consists of the characters of $A(\mathbb{A}_F)/A(\mathbb{A}_F)^1$, however the argument and the result hold for any family of idele class characters. In what follows, we use Tamagawa measures for all groups.

Let $\Phi \in \text{c-Ind}_{A(\mathbb{A}_F)^1 U(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(1)$, where c-Ind denotes compact induction, and write Φ in terms of its Mellin transform with respect to the left $A(\mathbb{A}_F)$-action:

$$\Phi(g) = \int_{\widehat{A(\mathbb{A}_F)/A(\mathbb{A}_F)^1}} f_{\chi\delta^{-1/2}}(g)\, d\chi$$

where $f_\chi \in I(\chi)$ and $d\chi$ is Haar measure on the unitary dual of $A(\mathbb{A}_F)/A(\mathbb{A}_F)^1$, note that the unitary dual of $A(\mathbb{A}_F)/A(\mathbb{A}_F)^1$ can naturally identified with the imaginary points $i\mathfrak{a}_{\mathbb{R}}^*$ of the Lie algebra of the dual torus via the exponential isomorphism.

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We can shift the contour of the integral for Φ and write

$$\Phi(g) = \int_{exp(\kappa+i\mathfrak{a}_{\mathbb{R}}^*)} f_\chi(g) d\chi$$

for any $\kappa \in \mathfrak{a}_{\mathbb{C}}^*$. In particular, we can shift the domain of convergence of the Eisenstein sum and then we will have:

$$\sum_{\gamma\in B(F)\backslash G(F)} \Phi(\gamma g) = \int_{exp(\kappa+i\mathfrak{a}_{\mathbb{R}}^*)} E(f_\chi, g)\ d\chi$$

as a function of rapid decay on the automorphic quotient $G(F)\backslash G(\mathbb{A}_F)$. We then integrate over $H(F)\backslash H(\mathbb{A}_F)$

$$\int_{H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma\in B(F)\backslash G(F)} \Phi(\gamma h)\ dh =$$

$$= \sum_{\xi\in[B(F)\backslash G(F)/B(F)]} \int_{H_\xi(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{H_\xi(F)\backslash H_\xi(\mathbb{A}_F)} \int_{exp(\kappa+i\mathfrak{a}_{\mathbb{R}}^*)} f_\chi(\xi a h) d\chi\ da\ dh$$

here $[B(F)\backslash G(F)/H(F)]$ denotes a finite set of representatives in $G(F)$ for the $(B(F), H(F))$-double cosets, and $H_\xi := H \cap \xi^{-1}B\xi$, similarly we will denote $B_\xi := B \cap \xi H \xi^{-1}$ and we will let Y denote the B-orbit of ξH on G/H.

For a fixed $h \in H(\mathbb{A}_F)$, the inner integral

$$\int_{H_\xi(F)\backslash H_\xi(\mathbb{A}_F)} \int_{exp(\kappa+i\mathfrak{a}_{\mathbb{R}}^*)} f_\chi(\xi a h) d\chi da$$

is equal to

$$\mathrm{vol}(B_\xi(F)\backslash B_\xi(\mathbb{A}_F)^1) \cdot \int_{B_\xi(\mathbb{A}_F)^1\backslash B_\xi(\mathbb{A}_F)} \int_{exp(\kappa+i\mathfrak{a}_{\mathbb{R}}^*)} f_\chi(a\xi h) d\chi da$$

by abelian Fourier analysis, the last expression is equal to

$$\int_{\delta^{-1/2}\eta_Y^{-1} exp(i\mathfrak{a}_{\mathbb{R}}^*)} f_\chi(\xi h)\ d\chi$$

where we have taken into account that $exp(\mathfrak{a}_Y^*)$ where $\mathfrak{a}_Y^*$ is the Lie algebra of A_Y^*, considered as a subgroup of the group of characters $exp(\mathfrak{a}^*)$ of $A(\mathbb{A}_F)/A(\mathbb{A}_F)^1$, is the orthogonal complement of the image of $B_\xi(\mathbb{A}_F)$.

To determine the most continuous part of the H-period, it is enough to consider those ξ which correspond to the orbits Y of maximal rank, we can move the contour of integration, this time to $exp(\kappa_Y + i\mathfrak{a}_{Y,\mathbb{R}}^*)$, where κ_Y is deep in the region so that the morphisms $\Delta_{\chi,v}^Y$ are convergent.

We can interchange the order of integration to express the contribution of the orbit Y as

$$\int_{exp(\kappa_Y+i\mathfrak{a}_{Y,\mathbb{R}}^*)} \int_{H_\xi(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} f_\chi(\xi h)\ dh d\chi$$

and the new inner integral is equal to $\prod_v \Delta_{\chi,v}^{Y,Tam}$, where the exponent "Tam" stands to show that we are using the Tamagawa measures, rather than the usual measure used in the paper [Sak13] to derive the formula for the spherical functions.

The following lemma tells us how $\Delta_{\chi,v}^{Tam}$ and Δ_χ are related

Lemma 2.1. *We have:*

$$\Delta_{\chi,v}^{Tam} = Q_v \Delta_{\chi,v}$$

Fix a finite set of places S, including the infinite ones and those finite places where our assumptions on the spherical variety $X = H\backslash G$ do not hold, such that we have a factorization: $f_\chi = \prod_v f_{\chi,v}$ where $f_{\chi,v}$ being the standard K_v-invariant functions: $f_{\chi,v}^0(bk) = \chi\delta^{1/2}(b)$ (this is usually denoted by $\phi_{K,\chi}$).

From the previous lemma 2.1, we have

$$\Delta_{\chi,v}^{Tam}(f_{\chi,v}^0) = Q_v\Delta_{\chi,v}(f_{\chi,v}^0) = \prod_{\check{\alpha}>0} \frac{1-q^{-1}e^{\check{\alpha}}}{1-e^{\check{\alpha}}}(\chi)\Omega_{\chi,v}(x_0) = \prod_{\check{\alpha}>0} \frac{1-q^{-1}e^{\check{\alpha}}}{1-e^{\check{\alpha}}}(\chi) \cdot L_{\chi,v}^{\frac{1}{2}}$$

recall that $L^{\frac{1}{2}}_{X,v}(\chi) = c_v \cdot \beta_v(\chi)$ where c_v is a quotient of products of local values for the Dedekind zeta function of F and β_v is a quotient of products of Dirichlet L-values which depend on χ. If we consider the product $\prod_{v \notin S} c_v$ it may not converge in general. However, we can make sense of it by considering the leading term of its Laurent expansion, when considered as a specialization of a product/quotient of translates of ζ^S. We should denote $(c^S)^*$ whenever c^S formally appears in the product, similarly we will denote $(L_X^{\frac{1}{2},S})^* = (c^S)^* \cdot \prod_{v \notin S} \beta_v(\chi)$. Therefore we get

Theorem 2.2. *The period integral of*

$$\sum_{\gamma \in B(F)\backslash G(F)} \Phi(\gamma g) = \int_{exp(\kappa + i\mathfrak{a}^*_{\mathbb{R}})} E(f_\chi, g)\ d\chi$$

over $H(F)\backslash H(\mathbb{A}_F)$ is equal to

$$\int_{exp(\kappa + i\mathfrak{a}^*_{X,\mathbb{R}})} (L_X^{\frac{1}{2},S})^* \sum_{[W/W_{(X)}]} (\tilde{j}^S_\omega(\chi) \prod_{v \in S} \Delta^{Y,Tam}_{\omega_{\chi},\ v}(f_{\omega_{\chi},v})\ d\chi$$

plus terms which depend on the restriction of f_χ, as a function of χ, to a subvariety of smaller dimension. Here $[W/W_{(X)}]$ denotes a set of representatives of minimal length for $W/W_{(X)}$-cosets, $\kappa \in \rho_{(X)} + \mathfrak{a}^*_{X,\mathbb{C}}$ is deep in the domain of convergence of Δ_χ, and f_χ the Mellin transform of Φ with respect to the normalized $A(\mathbb{A}_F)/A(\mathbb{A}_F)^1$-action, is assumed to be factorizable with factors $f^0_{\chi,v}$ for $v \notin S$.

3. Regularized periods of Eisenstein series

3.1. **Introduction.** Let G be a reductive group over a number field F, and let $H \subset G$ be a "nice" subgroup (e.g. spherical), then the following *period integral*

$$\Pi^H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})^1} \varphi(h)\ dh$$

converges absolutely for any cusp form φ on $G(\mathbb{A})$. The first goal of the JLR paper is to develop a method for defining and computing $\Pi^H(\varphi)$ for φ a more general automorphic form such as an Eisenstein series, in this case, the integral need not converge and we have to define it by means of a regularization procedure. Their second goal is to use this regularized period to obtain explicit formulas for the convergent period $\Pi^H(\Lambda^T E)$ for $\Lambda^T E$ a truncated Eisenstein series.

Let us recall some motivation, the periods $\Pi^H(\Lambda^T E)$ are of interest because they appear in the relative trace formula as a role analogous to that played by inner product of truncated Eisenstein series in the Arthur-Selberg trace formula. They arise when one computes the contribution from the most continuous spectrum of the relative trace formula. The RTF provides a general tool for studing distinguished representations. In many cases, it should be possible to characterize the H-distinguished cuspidal representations as images with respect to a functorial transfer to G from a third group G', general result of this type should eventually follow from a comparison of suitable relative trace formulas on G and G'.

Let's now describe their results in greater detail. They define a regularized period $\Pi^H(\varphi)$ for $H \subset G = \mathrm{Res}_{E/F} H$ with E/F quadratic using Arthur's truncation operators. They used a mixed truncation operator Λ^T_m which is a intermediate truncation between the truncation on G and the truncation on H. This truncation is best suited to the study of period integrals. In the next step, they computed the period of a truncated Eisenstein series

$$\int_{H(F)\backslash H(\mathbb{A})^1} \Lambda^T_m \varphi(h)\ dh$$

in terms of the regularized periods of the constant term of φ.

In their last two sections, they calculated more explicit result for $H = GL_n/F$. They obtain an explicit formula expressing $\Pi^H(\Lambda^T_m E(g, \varphi, \lambda))$ for any cuspidal Eisenstein series in terms of certain linear functionals $J(\xi, \varphi, \lambda)$ which they call *intertwining periods.*

The name "intertwining period" comes from the following reasons: first the map $\varphi \to J(\xi, \varphi, \lambda)$ is $H(\mathbb{A})$-invariant functional on $\mathrm{Ind}^G_P(\sigma \otimes e^\lambda)$, hence by Frobenious reciprocity, defines an intertwining operator. Furthermore, the J-functionals have several properties in common with the standard intertwining operators, their explicit formula for the period integral of truncated Eisenstein series is analogous to Langlands' formula for the inner product of cuspidal Eisenstein series. Finally, like the standard intertwining operators, the J-functionals can be merimorphically continued and satisfy a set of functional equations. The functional equations take the form

$$J(\xi, \varphi, \lambda) = J(s\xi s^{-1}, M(s, \lambda)\varphi, s\lambda)$$

For example in the $n = 2$ case, the functional equation reduces to

$$J(\xi, \varphi, \lambda) = m(\xi, \lambda)J(\xi, \varphi, -\lambda)$$

where $m(\xi, \lambda) = L(\lambda, 1_E)/L(\lambda + 1, 1_E)$.

The regularization of integrals is also useful in providing a more conceptual approach to the formulas for inner product of truncated Eisenstein series that occur in the Arthur-Selberg trace formula. Then regularized integral can be used to derive a formula for the convergent integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T \varphi(g)\ dg$$

apply to the case $G \subset G \times G$, the resulting formula can be viewed as a generalization of Langlands' formula for the inner product of truncated cuspidal Eisenstein series.

3.2. **The regularized period.** In this section, we will assume $H = GL_n/F$ and $G = \mathrm{Res}_{E/F}GL_n$ for E/F a quadratic extension of number fields.

The next theorem describes the regularized period of a cuspidal Eisenstein series. We will write $x \mapsto \overline{x}$ the conjugation of E over F and if π is a representation of $G(\mathbb{A}_E)$ or a Levi subgroup, we will write $\overline{\pi}$ for the representation $g \mapsto \pi(\overline{g})$. Let P be the parabolic subgroup of H corresponding to the partition $(n_1, \cdots, n_r)$ of n. Assume P is a proper subgroup if $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a representation of $M_E(\mathbb{A}_E)^1$ and $\lambda \in \mathfrak{U}_P^*$, we write $\sigma(\lambda)$ for the representation that extends σ to $M_E(\mathbb{A}_E)$ by $\sigma(\lambda)(am) = e^{\langle \lambda, H_{P_E}(a)\rangle}\sigma(m)$, we will write σ^* for the contragredient of σ.

Theorem 3.1. *Let $\varphi \in \mathcal{A}_P(G)_\sigma$ where $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a cuspidal representation of $M_E(\mathbb{A}_E)^1$ and let $E(\varphi, \lambda) = E(g, \varphi, \lambda)$ be the associated Eisenstein series, suppose $E(\varphi, \lambda)$ is regular at $\lambda = \lambda_0$ and $\Pi^{G/H}(E(\varphi, \lambda_0))$ is defined and non-zero. Then either $r = 1$ and $\varphi = E(\varphi, \lambda_0)$ is a distinguished cusp form on $G(\mathbb{A}_E)$ or $r = 2$ and $\sigma_2^* = \overline{\sigma_1}$. In this case, if $\langle Re\ \lambda,\ \alpha^\vee\rangle >> 0$ for the unique root $\alpha \in \Delta_P$, then the period is given by the following absolutely convergent integral*

$$\Pi^{G/H}(E(\varphi, \lambda)) = \int_{H_\eta(F)\backslash H(\mathbb{A})^1} e^{\langle \lambda + \rho_P, H_{P_E}(\eta h)\rangle}\ \varphi(\eta h)\ dh$$

where $\eta\overline{\eta}^{-1} = \xi$ is the unique element of $\Omega(M, M)$, furthermore $\Pi^{G/H}(E(\varphi, \lambda))$ extends to a meromorphic function of λ.

Proposition 3.2. *Let $\xi = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}$, we have $M_\eta = \{m = diag(g, \overline{g})\ :\ g \in GL_m(E)\ \}$, for $\langle Re\ \lambda, \alpha^\vee\rangle >> 0$, we have $\Pi^{G/H}(E(\varphi, \lambda)) =$*

$$\int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda + \rho_P, H_{P_E}(\eta h)\rangle}\Big(\int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \varphi(m\eta h)\ dm\ \Big)dh$$

where the integral above is absolutely convergent.

3.3. **Intertwining periods.** Fix a Levi subgroup M of H and a cuspidal representation σ of $M_E(\mathbb{A}_E)$ trivial on A_{P_E}, for $\xi \in \Omega_2(M, M)$, $\varphi \in \mathcal{A}_P(G)_\sigma$ and $\lambda \in (\mathfrak{U}_{P,\mathbb{C}}^*)_\xi^-$, we have the following definition

Definition 3.3. The following *intertwining period* attached to ξ is well defined

$$J(\xi, \varphi, \lambda) = \int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle \lambda + \rho_P, H_{P_E}(\eta h)\rangle}\Big(\int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \varphi(m\eta h)\ dm\ \Big)dh$$

where $\eta \in G(E)$ is any element satisfying $\eta\overline{\eta} = \xi$, recall that $P_\eta = M_\eta N_\eta$, $H_\eta = \eta P \eta^{-1} \in P_\eta(\mathbb{A})$.

Example 3.4. Let the notation as in theorem 3.1, from proposition 3.2 we have

$$\Pi^{G/H}(E(\varphi,\lambda)) = \int_{H_\eta(F)\backslash H(\mathbb{A})} e^{\langle \lambda+\rho_P, H_{P_E}(\eta h)\rangle}\varphi(\eta h)\ dh = J(\xi,\varphi,\lambda)$$

where $H_\eta = \eta^{-1}M_\eta\eta$ and $M_\eta \in \{\mathrm{diag}(g,\overline{g})\ :\ g\in GL_m(E)\}$.

Now let's consider a local analogue of the intertwining period in the unramified case, assume $n=2m$ and let $P=MN$ be the standard parabolic subgroup of type (m,m), we use the notation as before, then ξ is a non-trivial element in $\Omega(M,M)$ and $\eta\in GL_n(E)$ satisfies $\eta\overline{\eta}^{-1}=\xi$, assume $\sigma=\sigma_1\otimes\sigma_2$ is a cuspidal representation of $M_E(\mathbb{A}_E)$ and such that $\sigma_2\cong\overline{\sigma}_1^*$, then there is a unique linear form L' on the space of σ invariant under $M_\eta(\mathbb{A})$ namely

$$L'(\varphi) = \int_{M_\eta(F)\backslash M_\eta(\mathbb{A})^1} \varphi(m)\ dm$$

for $j=1,2$, we choose an identification of σ_j with a restricted tensor product $\otimes'\sigma_{jv}$, this identification presupposes the choice of K_v-fixed vectors x_{jv} in the space of σ_{jv} for almost all v. Thus σ_{jv} is a representation of $GL_m(E_v)$ where $E_v=E\otimes F_v$, we set $\sigma_v=\sigma_{1v}\otimes\sigma_{2v}$, since $\sigma_{2v}\cong\overline{\sigma}_{1v}^*$, there exists a non-zero linear map

$$L'_v:\ \sigma_{1v}\otimes\sigma_{2v}\longrightarrow\mathbb{C}$$

invariant under M_η, it is also unique up to scalar multiples, we may assume $L'_v(x_{1v}\otimes x_{2v})=1$ for almost all v. Then for a suitable normalization of L', we have

$$L'(\varphi)=\prod_v L'_v(\varphi_v)$$

where φ corresponds to a pure tensor $\otimes\varphi_v$.

In the global theory we used unnormalized induction, it is more convenient to use the normalized induction for the local computation, we now have

$$J(\xi,\varphi,\lambda) = \int_{H_\eta(\mathbb{A})\backslash H(\mathbb{A})} e^{\langle\lambda, H_{P_E}(\eta h)\rangle} L(\varphi)(\eta h)\ dh$$

we have the *local intertwining period*

$$J_v(\xi,\varphi_v,\lambda) = \int_{H_\eta(F_v)\backslash H(F_v)} e^{\langle\lambda, H_{P_E}(\eta h_v)\rangle} L_v(\varphi_v)(\eta h_v)\ dh_v$$

for $\lambda\in(\mathfrak{U}_P^*)_\xi^-$, then we have the following factorization

$$J(\xi,\varphi,\lambda)=\prod_v J_v(\xi,\varphi_v,\lambda)$$

the global intertwining period is equal to the product of local intertwining periods, combined with example 3.4, we are able to calculate the regularized period of Eisenstein series.

For the rest of this section, we will assume that v is a finite place such that σ_{jv} is unramified for $j=1,2$, we will compute the value of $J_v(\xi,\varphi_v,\lambda)$, we will recall the definition of the Asai L-function of an unramified representation of $GL_m(E)$, here we view $GL(m)_E$ as a group over F. Its L-group is

$${}^LGL(m)_E = GL_m(\mathbb{C})\times GL_m(\mathbb{C})\rtimes \mathrm{Gal}(E/F)$$

where the non-trivial element $\sigma_{E/F}\in\mathrm{Gal}(E/F)$ acts on the connected component by interchanging the factors. Let $V=\mathbb{C}^m$ and let T be the automorphism of $V\otimes V$ sending $x\otimes y$ to $y\otimes x$, we identify $GL(V\otimes V)$ with $GL_{m^2}(\mathbb{C})$ and define

$$\rho_A:\ {}^LGL(m,E)\longrightarrow GL_{m^2}(\mathbb{C})$$

where $\rho_A(g\times h\times 1)=g\otimes h$ and $\rho_A(1\times1\times\sigma_{E/F})=T$, write $\omega_{E/F}$ for the character of ${}^LGL(m)_E$ obtained by pulling back the non-trivial character of $\mathrm{Gal}(E/F)$. We have the following equality

$$L(\sigma_v,s,\rho_A)L(\sigma_v,s,\rho_A\otimes\omega_{E/F}) = L(s,\sigma_v\times\overline{\sigma_v})$$

where $L(s,\sigma_v\times\overline{\sigma_v})$ is the Rankin-Selberg convolution of σ_v and $\overline{\sigma_v}$. We will denote ρ_A^* the contragredient representaton of ρ_A.

We now begin our computation for $J_v(\xi,\varphi_v,\lambda)$ for φ_v fixed by K_v, suppose σ_{1v} is the unramified constituent of $\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)}\chi$ where B_m is the standard upper-triangular Borel subgroup of $GL(m)$ and $\chi=(\chi_1,\cdots,\chi_m)$ is an m-tuple of unramified characters of E_v^*. Then σ_{2v} is the unramified constituent of $\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)}\chi^{-1}$, let χ^* be the character if the upper triangular Borel subgroup $B(E_v)$ of $G(E_v)$ defined by the n-tuple $(\chi_1,\cdots,\chi_m,\chi_1^{-1},\cdots,\chi_m^{-1})$, we identify $\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)}\chi\otimes\mathrm{Ind}_{B_m(E_v)}^{GL_m(E_v)}\chi^{-1}$ with $\mathrm{Ind}_{B(E_v)}^{M(E_v)}\chi^*$, for ψ in the space of $\mathrm{Ind}_{B(E_v)}^{M(E_v)}\chi^*$, set

$$L'_v(\psi)=\int_{B_\eta(F_v)\backslash M_\eta(F_v)}\psi(m)\ dm$$

We identify π_v with the unramified constituent of the induced representation $\Sigma_v=\mathrm{Ind}_{B(E_v)}^{G(E_v)}\chi^*$ and on the space of Σ_v the functional L_v can be written as

$$L_v(\varphi)(g)=\int_{B_\eta(F_v)\backslash M_\eta(F_v)}\varphi(mg)\ dm$$

where dm is the semi-invariant measure on $B_\eta\backslash M_\eta(F_v)$. The local intertwining period can be written as

$$\begin{aligned}J_v(\xi,\varphi,\lambda)&=\int_{H_\eta(F_v)\backslash H(F_v)}e^{\langle\lambda,\ H_{P_E}(\eta h)\rangle}\ L_v(\varphi)(\eta h)\ dh\\&=\int_{H_\eta(F_v)\backslash H(F_v)}\int_{B_\eta\backslash M_\eta(F_v)}e^{\langle\lambda,H_{P_E}(\eta h)\rangle}\ \varphi(m\eta h)\ dm\ dh\\&=\int_{B'_\eta(F_v)\backslash H(F_v)}e^{\langle\lambda,H_{P_E}(\eta h)\rangle}\ \varphi(\eta h)\ dh\end{aligned}$$

where $B'_\eta(F_v)=H(F_v)\cap\eta^{-1}B_E(E_v)\eta$.

We have the following theorem on the unramified computation

Theorem 3.5. *Assume $v\notin S$ and let φ_v be the essential vector, for a suitable normalization of measures we have*

$$J_v(\xi,\varphi_v,\lambda)=\frac{L(\lambda,\sigma_{1v},\rho_A^*)}{L(\lambda+1,\sigma_{1v},\rho_A^*\otimes\omega_{E/F})}$$

here the essential vector is the unique function φ_v in the space Σ_v which is right invariant under $GL_n(\mathcal{O}_v)$ where $\mathcal{O}_v$ is the ring of integers in E_v and satisfies $\varphi_v(e)=1$.

We first prove this theorem in the case v splits in E, then we may identify $G(E_v)$ with $GL_n(E_{\omega_1})\times GL_n(E_{\omega_2})$ where $\omega_1,\ \omega_2$ are the places of E dividing v, we have $\sigma_{jv}=\sigma_{j\omega_1}\otimes\sigma_{j\omega_2}$ where $\sigma_{1\omega_1}\cong\sigma_{2\omega_2}^*$ and $\sigma_{1\omega_2}\cong\sigma_{2\omega_1}^*$. Conjugation acts by $(x,y)\mapsto(y,x)$ and $H(F_v)$ is imbedded diagonally. We may take $\eta=(1,\xi)$, then $H_\eta(F_v)$ is the Levi factor of the parabolic subgroup $P=MN$ of type (m,m) and $B'_\eta(F_v)=B(F_v)\cap H_\eta(F_v)$, use the Iwasawa decomposition, $H(F_v)=B'_\eta(F_v)N(F_v)K_F$, we obtain

$$\begin{aligned}J_v(\xi,\varphi,\lambda)&=\int_{B'_\eta(F_v)\backslash H(F_v)}e^{\langle\lambda,H_P(\eta h)\rangle}\ \varphi(\eta h)\ dh\\&=\int_{N(F_v)}e^{\langle\lambda,H_P(\xi n)\rangle}\varphi_2(\xi n)\ dn\end{aligned}$$

in other words J_v coincides with the standard intertwining operator applied to the essential vector in $\mathrm{Ind}_{P_{\omega_2}}^{G_{\omega_2}}(\sigma_{1\omega_2}\otimes\sigma_{2\omega_2})$. By the Gindikin-Karpelevic formula, the integral is equal to

$$\frac{L(\lambda,\sigma_{1\omega_2}^*\otimes\sigma_{2\omega_2})}{L(\lambda+1,\sigma_{1\omega_2}^*\otimes\sigma_{2\omega_2})}=\frac{L(\lambda,\sigma_{1\omega_2}^*\otimes\sigma_{1\omega_1}^*)}{L(\lambda+1,\sigma_{1\omega_2}^*\otimes\sigma_{1\omega_1}^*)}$$

and this is equal to $\frac{L(\lambda,\sigma_{1v}^*,\rho_A)}{L(\lambda+1,\sigma_{1v}^*,\rho_A\otimes\omega_{E/F})}$.

Remark 3.6. The unramified computation here is exactly the unramified computation needed for proof of the Casselman-Shalika formula.

We now assume that v remains prime in E, we drop v from the notation and first consider the case $H = GL_2(F)$ and $G = GL_2(E)$ where E/F is a unramified extension of p-adic fields, we also assume $p \neq 2$.

Proposition 3.7. *For $T = H \cap \eta^{-1} B \eta$, and φ the essential vector in the unramified representation of $GL_2(E)$ with Langlands class*

$$\begin{pmatrix} q_E^{\lambda/2} & 0 \\ 0 & q_E^{-\lambda/2} \end{pmatrix}$$

normalize the measure by assigning measure 1 to the ring of integers $\mathcal{O}_F$, then we have

$$\int_{T\backslash H} \varphi(\eta h)\ dh = ||i||^{1/2} \frac{1 + q^{-\lambda-1}}{1 - q^{-\lambda}}$$

In general, we have the following proposition, which is the inert case of theorem 3.3

Proposition 3.8. *Let E/F be an unramified quadratic extension of p-adic fields and let σ_1 be an unramified representation of $GL_m(E)$, then for the unique K-invariant $\varphi \in I_P(\sigma_1 \times \overline{\sigma}_1^*)$, we have*

$$J(\xi, \varphi, \lambda) = \frac{L(\lambda, \sigma_1, \rho_A^*)}{L(\lambda + 1, \sigma_1, \rho_A^* \otimes \omega_{E/F})}$$

Proof. In the above notation, suppose that $\sigma_1 = \mathrm{Ind}_{B_m(E)}^{GL_m(E)} \chi$ where $\chi = (\chi_1, \cdots, \chi_m)$ where $\chi_i = |\cdot|_E^{\lambda_i}$, we can view φ as an element of $\mathrm{Ind}_{B_{nv}}^{G_{nv}}(\chi_1, \cdots, \chi_n, \chi_1^{-1}, \cdots, \chi_n^{-1})$, to compute

$$J_v(\xi, \varphi, \lambda) = \int_{B'_\eta(F_v)\backslash H(F_v)} e^{\langle \lambda, H_{B_E}(\eta h)\rangle}\ \varphi(\eta h)\ dh$$

we shall regard it as a local intertwining period for an Eisenstein series and reduce to the $n = 2$ case using the functional equation. Let $\xi' = (1, 2) \cdots (2n-1, 2n)$, then ξ' is a minimal involution and $\xi = \omega^{-1} \xi' \omega$ where ω is defined by $\omega(i) = 2i - 1$ and $\omega(i + n) = 2i$ for $i = 1, \cdots n$. Then ω has the reduced decomposition

$$\omega = (s_{2n-2})(s_{2n-4} s_{2n-3}) \cdots (s_4 \cdots s_n s_{n+1})(s_2 \cdots s_{n-1} s_n)$$

we observe we have the following local functional equation proved by rewriting the absolutely convergent integral

$$J_v(\xi, \varphi, \lambda) = J_v(\xi', M(\omega, \lambda)\varphi, \omega\lambda)$$

the right hand side can be written as a local intertwining period with respect to the group $GL_2 \times \cdots \times GL_2$ (m times) and induction from $(\chi_1, \chi_1^{-1} \cdots \chi_m, \chi_m^{-1})$, by proposition 3.7, we have

$$J(\xi', \varphi, \lambda) = \prod_{i=1}^n (1 - q_E^{-\lambda_i} q_F^{-\lambda})^{-1} (1 + q_E^{-\lambda_i} q_F^{-(\lambda+1)})$$

by the formula of Gindikin and Karpelevic, $M(\omega, \lambda)\varphi = c(\lambda)\varphi$ where

$$c(\lambda) = \prod_{1 \leq i < j \leq n} \frac{(1 - q_E^{-\lambda_i - \lambda_j} q_E^{-\lambda})^{-1}}{(1 - q_E^{-\lambda_i - \lambda_j} q_E^{-(\lambda+1)})^{-1}}$$

compare this with the explicit formula for the Asai L-function, we get our result. □

Remark 3.9. It will be interesting to give a more systematic formulation of the local functional equation for the unramified computation for Galois periods in general.

3.4. **Main theorem.**

Theorem 3.10. *Let $\varphi \in \mathcal{A}_{P_E}(G)_\sigma$ where σ is a cuspidal representation of $M_E(\mathbb{A}_E)$, let $E(g, \varphi, \lambda)$ be the associated Eisenstein series, then as a meromorphic function of λ*

$$\int_{H(F)\backslash H(\mathbb{A})^1} \Lambda_m^T E(h, \varphi, \lambda)\ dh$$

is equal to

$$\sum_{(Q,s) \in \mathcal{G}(P,\sigma)} v_Q \frac{e^{\langle (s\lambda)_Q, T\rangle}}{\prod_{\alpha \in \Delta_Q} \langle (s\lambda)_Q, \alpha^\vee\rangle} J(\xi_Q, M(s, \lambda)\varphi, (s\lambda)_{P'}^Q)$$

Proof. Since σ is cuspidal, the Eisenstein series $E_{Q_E}(g,\varphi,\lambda)$ vanishes unless Q contains an associate of P and we obtain

$$\begin{aligned}&\int_{H(F)\backslash H(\mathbb{A})^1} \Lambda_m^T E(g,\varphi,\lambda)\ dg\\ &= \sum_Q (-1)^{d(Q)-d(H)} \int^*_{Q(F)\backslash H(\mathbb{A})^1} E_{Q_E}(g,\varphi,\lambda)\hat{\tau}_Q(H_{Q_E}(g)-T)\ dg\end{aligned}$$

where $E^{Q_E}(g, M(s,\lambda)\varphi,\lambda)$ is the Eisenstein series on Q_E induced from the function $M(s,\lambda)\varphi$. We must therefore compute the integrals

$$\int^*_{Q(F)\backslash H(\mathbb{A})^1} E^{Q_E}(g, M(s,\lambda)\varphi, s\lambda)\ \hat{\tau}_Q(H_{Q_E}(g)-T)\ dg$$

for $s\in\Omega(P,Q)$, by definition is equal to an period $\Pi^{M_{Q_E}/M_Q}$ of E^{Q_E} which is an Eisenstein series for the parabolic subgroup $P_E\cap M_{Q_E}$ of the group M_{Q_E}, and

$$\int^*_{\mathfrak{U}_Q} e^{\langle (s\lambda)_Q,\ 2X\rangle}\hat{\tau}_Q(2X-T)\ dX \tag{3.1}$$

By theorem 3.1 applied to a product of linear group, we get $\Pi^{M_{Q_E}/M_Q}$ vanishes unless (Q,s) belongs to $\mathcal{G}(P,\sigma)$. If $(Q,s)\in\mathcal{G}(P,\sigma)$, then it is equal to the following intertwining period integral for the group M_Q:

$$J^{M_Q}(\xi_Q, M(s,\lambda)\phi)^{K_F}, (s\lambda)^Q_{P'})$$

this is the same as the following intertwining period for the group G

$$J(\xi_Q, M(s,\lambda)\phi, (s\lambda)^Q_{P'})$$

on the other hand, we have

$$\int^*_{\mathfrak{U}_Q} e^{\langle (s\lambda)_Q,\ 2X\rangle}\hat{\tau}_Q(2X-T)\ dX = v_Q\frac{e^{\langle (s\lambda)_Q, T\rangle}}{\prod_{\alpha\in\Delta_Q}\langle (s\lambda)_Q,\alpha^\vee\rangle}$$

□

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