THE DUAL GROUP OF A SPHERICAL VARIETY

RUI CHEN

1. The dual group of a spherical variety

1.1. **Introduction.** This is a study note for the Knop-Schalke's paper [KS17], they proved the existence of the adapted map as conjectured in [SV17] in great generality.

1.2. **Root systems.**

Proposition 1.1. *Let Σ be a subset of an Euclidean vector space V, assume that V is contained in some open half-space and*

$$\langle \sigma, \tau^\vee \rangle = \frac{2(\sigma,\tau)}{(\tau,\tau)} \in \mathbb{Z}_{\leq 0}$$

for all $\sigma \neq \tau \in \Sigma$, then Σ is the basis of a finite root system.

Let $(\Lambda, S, \Lambda^\vee, S^\vee)$ be the based root datum of the connected group G and $\alpha \mapsto \alpha^s$ be an involution on S, with ${}^s\alpha^\vee := ({}^s\alpha)^\vee$ we get an involution on $S^\vee$

Definition 1.2. The involution s is called a folding if for all $\alpha, \beta \in S$

- $\langle \alpha, {}^s\alpha^\vee \rangle = 0$ whenever $\alpha \neq {}^s\alpha$.
- $\langle \alpha - {}^s\alpha, \beta^\vee + {}^s\beta^\vee \rangle = 0$.

1.3. **Weak spherical datum.**

Definition 1.3. A *weak spherical datum* with respect to a root datum $\mathcal{R} = (\Lambda, S, \Lambda^\vee, S^\vee)$ is a triple (χ, Σ, S^p) where $\chi \subseteq \Lambda$ and $\Sigma \subseteq \chi$, $S^p \subseteq S$ are subsets such that the following axioms are satisfied

- For every $\sigma \in \Sigma$ there is a subset $|\sigma| \subseteq S$ such that
$$\sigma = \sum_{\alpha \in |\sigma|} n_\alpha \alpha$$
such that σ is a weak spherical root.
- $\langle \chi, \alpha^\vee \rangle = 0$ for $\alpha \in S^p$.
- Let $\sigma = \alpha + \beta \in \Sigma$ of type D_2 then $\langle \chi, \alpha^\vee - \beta^\vee \rangle = 0$.
- Let $\alpha, \beta \in S$ with $\alpha, \alpha + \beta \in \Sigma$, then $\langle \beta, \alpha^\vee \rangle \neq -1$.

The advantage of working with weak spherical datum is that one can attach weak spherical datum to any G-variety

Proposition 1.4. *Let X be a G-variety, then (χ, Σ, S^p) is a weak spherical datum where*

$$\Sigma := \{\sigma_{norm} \mid \sigma \in \Sigma(X)\}, \quad \chi := \chi(X) + \mathbb{Z}\Sigma$$
$$S^p := \{\ \alpha \in S \mid P_\alpha x = Bx \text{ for } x \text{ in a dense subset of } X\}$$

Lemma 1.5. *Let (χ, Σ, S^p) be a weak spherical datum and $\sigma, \tau \in \Sigma$ be distinct, then $(\sigma, \tau) \leq 0$.*

Following [SV17], we are going to associate certain roots of G to every spherical root $\sigma \in \Sigma$.

Lemma 1.6. *Let $\sigma \in \Sigma \backslash \Phi$, then there exists a unique set $\{\gamma_1,\ \gamma_2\}$ of positive roots such that*

- *$\sigma = \gamma_1 + \gamma_2$.*
- *γ_1 and γ_2 are strongly orthogonal.*
- *$\gamma_1^\vee - \gamma_2^\vee = \delta_1^\vee - \delta_2^\vee$ for $\delta_1, \delta_2 \in S$.*

Date: June 2024.

The roots γ_1, γ_2 will be called associated to σ

Definition 1.7. For $\sigma \in \Sigma$, we let

$$\sigma^\wedge := \begin{cases} \{\sigma^\vee\} & \sigma \in \Phi \\ \{\gamma_1^\vee,\ \gamma_2^\vee\} & \sigma, \gamma_1, \gamma_2 \text{ as in previous lemma} \end{cases}$$

1.4. **Existence of the adapted map.**

Theorem 1.8. *Let (χ, Σ, S^p) be a weak spherical datum, then $(\chi, \Sigma, \chi^\vee, \Sigma^\vee)$ is a based root datum.*

Proof. This follows from proposition 1.1 and lemma 1.5. □

Theorem 1.9. *Let $\mathcal{S} = (\chi, \Sigma, S^p)$ be a weak spherical datum, then there exists a unique connected reductive subgroup $G^\wedge_{\mathcal{S}} \subset G^\vee$ containing $T^\vee$ such that $\Sigma^\wedge$ is its set of simple roots.*

Definition 1.10. Let $\mathcal{S} = (\chi, \Sigma, S^p)$ be a weak spherical datum, a homomorphism $\varphi : G^\vee_{\mathcal{S}} \to G^\vee$ is called adapted if it factors through $G^\wedge_{\mathcal{S}} \subseteq G^\vee$ and if it compatible with the map on maximal tori

$$\begin{array}{ccc} A^\vee & \longrightarrow & T^\vee \\ \downarrow & & \downarrow \\ G^\vee_{\mathcal{S}} & \longrightarrow & G^\wedge_{\mathcal{S}} \end{array}$$

To show the existence of adapted homomorphisms we observe that the sets $\sigma^\wedge$ for $\sigma \in \Sigma$ partition $\Sigma^\wedge$ into subsets of size at most 2, hence there exists a unique involution acting on $\Sigma^\wedge$ with orbits $\sigma^\wedge$.

Lemma 1.11. *The action of s on $\Sigma^\wedge$ is a folding in the sense of definition 1.2.*

Theorem 1.12. *Let $\mathcal{S} = (\chi, \Sigma, S^p)$ be a weak spherical datum, then there exists an adapted homomorphism $\varphi : G^\vee_{\mathcal{S}} \to G^\vee$.*

This follows from a lemma on folding.

References

[KS17] Friedrich Knop and Barbara Schalke. The dual group of a spherical variety. *Transactions of the Moscow Mathematical Society*, 78:187–216, 2017.

[SV17] Yiannis Sakellaridis and Akshay Venkatesh. *Periods and harmonic analysis on spherical varieties*. Société mathématique de France, 2017.