DICHOTOMY FOR GENERIC SUPERCUSPIDAL REPRESENTATIONS OF G2

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1. Introduction

This is a study note for the paper [SW11].

Let k be a finite extension of $\mathbb{Q}_p$ for p a prime number, we work with the k-points of algebraic groups. In this paper, we prove a precise correspondence between the generic supercuspidal irreducible representations of the exceptional group G_2 and certain generic supercuspidal irreducible representations of the classical groups PGL_3 and $PGSp_6$. This correspodence is phrased as a *dichotomy.* Every generic supercuspidal irrep τ of G_2 we associate either a generic supercuspidal irrep σ of $PGSp_6$ whose spin L-function has a pole at $s = 0$ or a contragredient pair of generic supercuspidal irreps $\{\rho, \tilde{\rho}\}$ of PGL_3. Symbolically, we write this dichotomy as a function Δ

$$\Delta : \ \mathrm{Irr}_g(G_2) \longrightarrow \mathrm{Irr}_{g,Spin}(PGSp_6) \sqcup \frac{\mathrm{Irr}_g(PGL_3)}{\mathrm{contra}}$$

The main result is suggested by Langlands' conjectural parametrization of the generic supercuspidal irreps of G_2, PGL_3 and $PGSp_6$. The results on Langlands parameters depend essentially on the structure theory of the complex simple groups $\mathrm{G}_2(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{C})$ and $\mathrm{Spin}_7(\mathbb{C})$. For this reason, we demonstrate the precise dichotomy at the level of Langlands parameters in the first section.

The second section is devoted to the structure theory of certain algebraic groups over the p-adic field k, including the construction of exceptional groups and their parabolic subgroups.

The third section provides the definition of the dichotomy map Δ. Specifically, the dichotomy is realized via theta correspondence using the dual pairs $G_2 \times PGL_3 \subset E_6$ and $G_2 \times PGSp_6 \subset E_7$ and the minimal representations of E_6 and E_7. Such theta correspondence have been studied in the literature. We refine the results of Ginzburg-Rallis-Soudry who first considered the "tower of theta correspondence" for G_2. Using extensive analysis of Jacquet modules for the minimal representation of E_6 and E_7, we are able to demonstrate that this pair of theta correspondence determines a dichotomy function Δ.

The fourth section is devoted to proving the injectivity of the dichotomy map Δ, through a study of the Whittaker and Shalika functionals. When considering a generic supercuspidal irrep ρ of PGL_3, the fibre $\Delta^{-1}(\{\rho, \tilde{\rho}\})$ has cardinality at most equal to the dimension of a space of Whittaker functionals on ρ. The uniqueness of Whittaker functionals yields injectivity of the dichotomy map. However, when considering a generic supercuspidal irrep σ of $PGSp_6$, the fiber $\Delta^{-1}(\sigma)$ has cardinality equal to the dimension of a space of a Shalika functionals on σ. This subgroup is a cubic analog of the Shalika subgroup of GL_n. In this fourth section, we prove the uniqueness of the Shalika functionals for supercuspidal irreps of GSp_6. It almost immediately follows that the dichotomy map is injective.

The fifth section focuses on the set of generic supercuspidal irreps of $PGSp_6$ in the image of Δ, precisely those generic supercuspidal irreps of $PGSp_6$ with non-vanishing Shalika functional occur in this image. However Langlands conjecture predict another characterization of the image of dichotomy: a generic supercupidal irrep σ of $PGSp_6$ should occur in the image of dichotomy if and only if its degree-eight spin L-function has a pole at $s = 0$. One direction, that a non-vanishing of the Shalika functional implies that the L-function has a pole, requires an analysis of the minimal representation of E_8, the construction of Shahidi of the spin L-function and connections to reducibility points of F_4 parabolically induced from GSp_6. The other direction, that if $L(\sigma, \mathrm{Spin}, s)$ has a pole at $s = 0$ then σ has a nonvanishing Shalika functional, requires the Bump-Ginzburg integral representation of the Spin L-function, and global methods to demonstrate that the BP construction agrees with Shahidi's for the spin L-function.

Many non-generic representations of G_2 also arise from the inner form $PD^\times$ of PGL_3.

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2. Shalika functionals

2.1. **The Shalika subgroup.** We view GSp_6 as a group of symplectic similitudes, we let M_2 denote the abelian unipotent algebraic group of two-by-two matrices, if g is such a matrix, we write g^t its transpose. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$J_3 = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$$

Let GSp_6 be the algebraic group of symplectic similitudes

$$GSp_6 = \{g \in GL_6 \ : \ gJ_3g^t = \mathrm{sim}(g) \cdot J_3\}$$

the resulting character $\mathrm{sim}: \ GSp_6 \to GL_1$ is called the similitude character.

Let $Q_3 = L_3U_3$ be the maximal parabolic subgroup of GSp_6 with Levi component

$$L_3 = \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \det(g^{-1}h) \cdot g \end{pmatrix}, \ g, h \in GL_2$$

and unipotent radical U_3. The center of U_3 is of three dimensional

$$Z_3 = \{ \begin{pmatrix} I & 0 & Z \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \ : \ ZJ + JZ^t = 0\}$$

there is an isomorphism of unipotent groups $U_3/Z_3 \to M_2$ given by

$$\begin{pmatrix} I & X & Z \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \mapsto X$$

there is also an isomorphism of reductive groups $L_3 \to GL_2 \times GL_2$ given by

$$\begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \det(g^{-1}h) \cdot g \end{pmatrix} \mapsto (g, h)$$

If $g \in GL_2$, then $\Delta(g)$ is identified with an element of $L_3 \subset GSp_6$

$$\Delta(g) = \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}$$

we will write S for the "Shalika subgroup"

$$S = \Delta(GL_2) \ltimes U_3 \subset Q_3$$

Define a character ψ_3 of U_3 by

$$\psi_3(u) = \psi_k(-\mathrm{Tr}(X))$$

for a matrix $u \in U_3$ projecting to $X \in M_2$, $\Delta(GL_2)$ is precisely the centralizer of the character ψ_3 in L. Hence the character ψ_3 can be extended uniquely to a character ψ_S of S such that $\psi_S(\Delta(g)) = 1$ for all $g \in GL_2$.

When σ is a smooth representation of GSp_6, we define the space of *Shalika functionals* by

$$\mathrm{Sh}(\sigma) = \mathrm{Hom}_S(\sigma, \psi_S)$$

this is very similar to the Shalika functionals considered by Jacquet and Rallis.

2.2. **Theta correspondence.** The importance of Shalika functionals in the theta correspondence is the following

Lemma 2.1. *Suppose σ is a generic supercuspidal irrep of $PGSp_6$, then there is a linear isomorphism*

$$Wh_{G_2}(\Theta(\sigma)) = \Theta(\sigma)_{N_2,\psi_2} \cong Sh(\sigma)$$

Proof. Let N_2 be the unipotent radical of a Borel subgroup of G_2 and ψ_2 a principal character of N_2, let $Q_2 = L_2U_2$ be a maximal parabolic subgroup of G_2 such that U_2 is contained in N_2 and N_2/U_2 corresponds to a short simple root.

Then $Wh_{G_2}(\Pi_7) = (\Pi_7)_{N_2,\psi_2}$ can be computed in two stages

$$(\Pi_7)_{N_2,\psi_2} = ((\Pi_7)_{U_2,\psi_2})_{N_2,\psi_2}$$

This was done by previous paper of Savin and Gross.

Let $S^\circ \subseteq S$ be the semidirect product of GL_2 with $U_3^\circ \subseteq U_3$ where U_3° contains the center Z_3 and U_3°/Z_3 corresponds to trace zero matrices in U_3/Z_3, then

$$(\Pi_7)_{U_2,\psi_2} \cong c - \mathrm{Ind}_{S^\circ}^{GSp_6}(\mathbb{C})$$

Under this identification the action of N_2 on $(\Pi_7)_{U_2,\psi_2}$ can be identified with the action of S/S° by left translation on $c - \mathrm{Ind}_{S^\circ}^{GSp_6}(\mathbb{C})$, this implies that

$$Wh_{G_2}(\Pi_7) = (\Pi_7)_{N_2,\psi_2} \cong c - \mathrm{Ind}_{S}^{GSp_6}(\psi_S)$$

as a representation of GSp_6.

Applying $\mathrm{Hom}_{GSp_6}(\sigma, \cdot)$ on both side we get

$$Wh_{G_2}(\Theta(\sigma)) = \Theta(\sigma)_{N_2,\psi_2} \cong Sh(\sigma)$$

□

Since $\Theta(\sigma)$ is multiplicity free and supercuspidal, and every subrepresentation is generic, we get the following proposition

Proposition 2.2. *Suppose that σ is a generic supercuspidal irrep of $PGSp_6$ then $\Theta(\sigma)$ is non-zero if and only if $Sh(\sigma) \neq 0$.*

3. L-function and periods

Theorem 3.1. *Let σ be a generic supercuspidal of $PGSp_6$, then the following three conditions are equivalent.*

(1) σ has a non-vanishing Shalika functional.
(2) Shahidi's L-function $L(\sigma, Spin, s)$ has a pole at $s = 0$.
(3) The Bump-Ginzburg-Vo L-function $L(\sigma, Spin, s)$ has a pole at $s = 0$.

References

[SW11] Gordan Savin and Martin H Weissman. Dichotomy for generic supercuspidal representations of G_2. *Compositio Mathematica*, 147(3):735–783, 2011.