DERIVED STRUCTURES IN THE DUALITY OF AUTOMORPHIC PERIODS

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1. Introduction

This is a study note for Eric Chen's thesis [Che23], some examples of unfolding of the global periods have been studied.

2. Automorphic periods

Let $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ be an additive character whose conductor of ψ at each place v is even. That is to say, there exists for each v, an even integer $2m_v$ with the property that ψ is trivial on $\varpi_v^{-2m_v}\mathcal{O}_v$.

We write

$$\partial^{1/2} = (\varpi_v^{m_v}) \in \mathbb{A}^\times$$

Definition 2.1. Let k be a field and let X be an affine k-variety with $\mathbf{G}_m$-action, we say that X is conical if the coordinate ring $k[X]$ has only nonnegative $\mathbf{G}_m$-weights and the 0th graded piece is isomorphic to k.

For X an affine conical spherical variety, and such that T^*X is hyperspherical, we consider the following unitarily-normalized action of the adelic points of G on the space of adelic Schwartz functiosn $\mathcal{S}(X(\mathbb{A}_F))$

$$g * \Phi(x) = |\eta(g)|^{1/2}\Phi(xg)$$

where η is the eigencharacter on the smooth local X^{sm}.

To form the normalized theta series on X, we want to use not the standard characteristic function of integral points on $X(\mathbb{A}_F)$ but its translate through $\partial^{1/2}$

$$\Phi(x) := \text{right translate of } \Phi^0 \text{ through } \partial^{1/2}$$

let $\mathring{X} \subset X$ be the open subvariety, the normalized theta series of X on $G(\mathbb{A})$ is defined by multiplying the Poincare series $\sum_{x\in\mathring{X}(F)}(g, \partial^{1/2}) * \Phi(x)$ by a unitary normalization factor

$$\theta_X(g) = \Delta^{\frac{\dim X - \dim G}{4}} |\partial^{1/2}|^{1/2} \sum_{x\in\mathring{X}(F)} g * \Phi(x) \tag{2.1}$$

Using the theta series we can define the *normalized automorphic X-period* for an unramified automorphic form f on G by

$$P_X(f) := \int_{[G]} \theta_X(g) f(g)\, dg$$

Lemma 2.2. *Suppose X is a conical G-variety with respect to a central $\boldsymbol{G}_m \subset G$ such that $G/\boldsymbol{G}_m$ is semisimple, and take $\mathring{X} = X - \{0\}$, then $\int_{[G]} \theta_X(g)f(g)\, dg$ is absolutely and uniformly convergent if f is a cusp form.*

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3. Unfolding to the Whittaker model

3.1. Hecke periods. For the Hecke period, we choose a point $x_0 \in X(k)$, then we have $X = H\backslash G$ with $H = G_{x_0}$, the theta series is

$$\theta_X(g) = \sum_{\gamma \in H\backslash G(F)} \Phi(\gamma g)$$

hence for $\Phi = \delta_{x_0}$

$$P_X(f) = \int_{[G]} \theta_X(g) f(g)\ dg = \int_{[T]} f(g)\ dg$$

and we have

$$\begin{aligned} P_X^{\text{norm}}(f) &= \Delta^{-1/4} \int_{[T]} f(t)\ dt \\ &= \Delta^{-1/4} \int_{T(\mathbb{A})} W_f(t) dt \\ &= \prod_v P_{X,v} \end{aligned}$$

where $P_{X,v} = \int_{T(F_v)} W_v^{un}(t_v a_{0,v}^{-1})\ dt = L_v(\frac{1}{2}, f)\chi_v(\varpi_v^{-m_v})$.

3.2. Smooth Rankin-Selberg case. For $G = GL_n^2$, $X = GL_n \times \mathbb{A}^n$, consider the base point

$$\omega = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \dots & & \\ (-1)^{n-1} & & & \end{pmatrix}$$

the action is given by

$$(x, v)(g_1, g_2) = (g_1^T x g_2, v g_2)$$

We can compute the stabilizer of $(\omega \times 0)$ on $GL_n \times \mathbb{A}^2$

$$GL_n^{\tau\Delta} := \mathrm{Stab}_{GL_n^2}(\omega, 0) = (g, g^\tau)$$

as

$$g^T \cdot \omega \cdot g^\tau = g^T \cdot \omega \cdot \omega (g^T)^{-1} \omega^{-1} = (-1)^{n-1} \omega^{-1} = \omega$$

Let $\Phi^0 = \Phi_1 \otimes \Phi_2$ be the standard Schwartz function on $X = GL_n \times \mathbb{A}^n$, then $\mathring{X}(F) = GL_n(F) \times (F^n - 0)$, and from (2.1) we get

$$\theta_X(g_1, g_2) = (\cdots) \sum_{x \in GL_n(F),\ y \in F^n - 0} \Phi_1(g_1^T x g_2) \Phi_2(y g_2 \partial^{1/2})$$

We can compute the integral $\langle \theta_X,\ f_1 \times f_2 \rangle$ as

$$\begin{aligned} \langle \theta_X, f_1 \times f_2 \rangle &= (\cdots) \int_{GL_n^{\tau\Delta}(F)\backslash GL_n(\mathbb{A})^2} \Phi_1(g_1^T \omega g_2) f_1(g_1) f_2(g_2) \sum_{y \in F^n - 0} \Phi_2(y g_2 \partial^{1/2}) |g_2|^{1/2} \\ &= (\cdots) \int_{GL_n(F)\backslash GL_n(\mathbb{A})} f_1(g^\tau) f_2(g) \sum_{y \in F^n - 0} \Phi_2(y g \partial^{1/2}) |g|^{1/2} \\ &= (\cdots) \int_{P_n(F)\backslash GL_n(\mathbb{A})} f_1^\tau(g) f_2(g) \Phi_2(y g \partial^{1/2}) |g|^{1/2}\ dg \end{aligned}$$

now unfold $f_i(g)$ via $f_i(g) = \sum_{\gamma \in N\backslash P_n(F)} W_i(\gamma g)$ with the Whittaker functions W_i of f_i being normalized using probablity measures on the unipotent subgroups.

Then the above becomes

$$(\cdots) \int_{N_F\backslash GL_n(\mathbb{A})} W_1^\tau(g) W_2(g) \Phi(y g \partial^{1/2}) |g|^{1/2}\ dg$$

we may replace the domain of integration by $N(\mathbb{A})\backslash GL_n(\mathbb{A})$. In other words, the above can be written as an Euler-factorizable expression

$$(\cdots)\int_{N(\mathbb{A})\backslash GL_n(\mathbb{A})} W_1^\tau(g)W_2(g)\Phi(yg\partial^{1/2})|g|^{1/2}\ dg$$

we now proceed with the calculation of the local factors

$$(\cdots)\int_{N_v\backslash GL_n(F_v)} W_{1,v}^\tau(g)W_{2,v}(g)\Phi(yg\partial_v^{1/2})|g|_v^{1/2}\ dg_v$$

Use the Iwasawa decomposition to write the integral over A_v, then we have

$$(\cdots)\prod_v\int_{A_v} W_1^{un,\tau}(a_0^{-1}a)W_2^{un}(a_0^{-1}a)e^{-2\rho}(a)|e^{-2\rho}(a)|\Phi(y\partial^{1/2}a)|a|^{1/2}\ da$$

After a local unramified computation, we get

$$(\cdots)\prod_v\int_{A_v} W_1^{un,\tau}(a)W_2^{un}(a)|e^{-2\rho}(a)|\Phi(ya)|a|^{1/2}da = (\cdots)\prod_v L_v(\frac{1}{2}, f_1^\tau\times f_2)$$

References

[Che23] Eric Y Chen. *Derived Structures in the Duality of Automorphic Periods.* PhD thesis, Princeton University, 2023.