CATEGORICAL PLANCHEREL FORMULA

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1. Introduction

This is my note for Akshay Venkatesh's lecture on categorical Plancherel formula- Plancherel formula as formulated in the sheaf language.

2. Explicit Plancherel formula

In this section, we will assume $F = \mathbb{F}_q((t))$ and $K = PGL_2(\mathcal{O}) \subset G = PGL_2(F)$. Let $\mathscr{T} = G/K$, this $\mathscr{T}$ has a structure of $q+1$-valent tree.

Let T_n be the Hecke operator corresponding to the $n+1$-dimensional representation of the dual group SL_2, it acts on the functions on $\mathscr{T}$ by

$$T_n f(x) = \sum_y f(y)$$

where we sum over y whose distance to x lies in $\{n,\ n-2,\ n-4, \cdots\}$.

Using the Cartan decomposition, we can view $f \in K\backslash\mathscr{T}$ as $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, the inner product will be taken as $\langle f, g\rangle = \sum_{x\in\mathscr{T}} f(x)\overline{g(x)}$.

Using the formula for T_n, we can compute that $\langle T_n\delta_0, \delta_0\rangle = 1$ if n is even and $= 0$ if n is odd.

From this we deduce that there is a unique measure μ on the conjugacy classes in SU_2 such that

$$\langle \frac{T_n}{q^{n/2}}\delta_0, \delta_0\rangle = \int \chi_n \ d\mu$$

where χ_n is the character of the $n+1$ dimensional irreducible representation and up to a constant factor, μ equals to (q-character of $\mathfrak{sl}_2\times$ Haar measure).

Here if g is an automorphism of an affine complex algebraic variety Y, commuting with a $\mathbb{G}_m$-action on Y, we can define its q-character as

$$\chi_q(g) = \text{" trace of} g \times q^{-1/2} \text{ on } \mathbb{C}[Y]\text{"} = \sum_i q^{-i/2}\text{trace}(g|\mathbb{C}[Y]_i)$$

where $\mathbb{C}[Y]_i$ is the i-th graded piece for the $\mathbb{G}_m$ action on $\mathbb{C}[Y]$.

Example 2.1. We have

- When $x \in \mathbb{C}^*$ acts on $Y = T^*\mathbb{A}^1$ its q-character is
$$\frac{1}{(1-q^{-1/2}x)(1-q^{-1/2}x^{-1})}$$
- when $g = \begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}$ acts on $Y = T^*\mathbb{A}^2$, its q-character is
$$\frac{1}{(1-q^{-1/2}x)^2(1-q^{-1/2}x^{-1})^2}$$
- when $g \in SL_2(\mathbb{C})$ as above act via the adjoint action on $Y = \mathfrak{sl}_2$ with the action of $\lambda \in \mathbb{G}_m$ to scale to λ^2, its q-character is
$$\frac{1}{(1-q^{-1})(1-q^{-1}x^2)(1-q^{-1}x^2)}$$

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3. Categorical Plancherel formula

Let k be a local field and G a connected reductive group over k, let X be a variety with G-action, we assume that $M = T^*X$ is hyperspherical, and hence its hyperspherical dual $\check{M}$ is defined, we have an explicit unramified Plancherel formula for X of the form (generalize the explicit Plancherel in the previous section)

$$\langle T_V\delta_0,\ T_W\delta_0\rangle = \int_{\check{G}_{compact}} \chi_V \overline{\chi}_W \ (q - \text{character of } \check{M})$$

we have

$$\begin{aligned} RHS &= \int \chi_V\overline{\chi}_W(\sum q^{-\frac{i}{2}}\chi_{\mathbb{C}[M]_i}) \\ &= \sum q^{-\frac{i}{2}} \dim \mathrm{Hom}_{\check{G}}(W, V\otimes \mathbb{C}[\check{M}]_i) \\ &= \mathrm{trace}(q^{-\frac{1}{2}} \text{ on } \mathrm{Hom}_{\check{G}}(W, V\otimes\mathbb{C}[\check{M}]_i)) \end{aligned}$$

we denote $\underline{W} = W\otimes\mathbb{C}[\check{M}]$, then the final trace can also be written as

$$\mathrm{trace}(q^{-\frac{1}{2}} \text{ on } \mathrm{Hom}(\underline{W}, \underline{V}))$$

Now let's assume $F = \mathbb{F}_q((t))$, $X_F/G_{\mathcal{O}}$ is the $\mathbb{F}_q$-points of some "reasonable" algebraic variety/stack, we are going to geometrize the Hecke action on $C_c(X_F)^{G_{\mathcal{O}}}$ to action of Hecke category on $Shv(X_F/G_{\mathcal{O}})$. Let's denote δ_0 the trace function $\mathbb{Q}_\ell|_{X(\mathcal{O})} = \underline{\delta_0}$. Now we can use the sheaf-function dictionary to rewrite

$$\mathrm{trace}(q^{-\frac{1}{2}} \text{ on } \mathrm{Hom}(\underline{W}, \underline{V}))$$

as

$$\mathrm{tr}(\mathrm{Fr}^{-1}|\ \mathrm{Hom}(\underline{T_W} * \underline{\delta_0}, \underline{T_V} * \underline{\delta_0})$$

which is equal to

$$\int \chi_V\overline{\chi_W}\ (q - \text{character of } \check{M}) = \mathrm{Hom}_{\check{M}/\check{G}}(\underline{W}, \underline{V})$$

The matching of boundary conditions predict that we have the following equivalence of categories: for $F = \mathbb{C}((t))$ (topological version) or $\overline{\mathbb{F}_q}((t))$ (arithmetic version)

$$\begin{aligned} Shv(X_F/G_{\mathcal{O}}) &\longrightarrow QC(\check{M}/\check{G}) \circlearrowright \mathbb{G}_{gr} \\ \underline{\delta_0} &\longmapsto \mathcal{O}_{\check{M}} \end{aligned}$$

here $Shv(X_F/G_{\mathcal{O}})$ is the boundary condition produced by X, and the $\mathbb{G}_{gr}$ action is introduced to get the correct q-character, it in fact corresponds to the evaluation point of the local L-function.

We can obtain the following corollary of the previous conjecture:

$$\mathrm{Hom}_{X(F)/G(\mathcal{O})}(\underline{\delta_0}, \underline{\delta_0}) = \mathrm{Hom}_{\check{M}/\check{G}}(\mathcal{O}, \mathcal{O})$$

over $\mathbb{C}((t))$, the left handside gives us $H^*_{G(\mathbb{C})}(X(\mathbb{C}))$, the $G(\mathbb{C})$-equivariant cohomology.

Example 3.1. For $X = \mathbb{A}^1$, $G = GL_1$, we have $\check{G} = GL_1$ and $\check{M} = T^*\mathbb{A}^1$, the $G(\mathbb{C})$-equivariant cohomology of $X(\mathbb{C})$ is $\mathbb{C}[[\xi]] = H^*(BS^1)$, and for $\check{G} = GL_1$, $\check{M} = T^*\mathbb{A}^1$, we have

$$\lambda\cdot(x, y) = (\lambda x, \frac{y}{\lambda})$$

hence $\mathbb{C}[\check{M}]^{\check{G}} = \mathbb{C}[xy]$.

Example 3.2. $X = \mathbb{G}_m\backslash PGL_2$, $G(\mathbb{C})$-equivariant cohomology of X is $\mathbb{C}[\xi_2]$. For the dual side, $\check{G} = SL_2$, $\check{M} = T^*\mathbb{A}^2 \cong \mathrm{Mat}_2$

$$\mathbb{C}[\check{M}]^{\check{G}} = \mathbb{C}[\mathrm{Mat}_2]^{SL_2} = \mathbb{C}[\det]$$