BOREL-TITS THEORY

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1. Introduction

This is my study note for the Borel-Tits theory following Brain Conrad's course notes `https://virtualmath1.stanford.edu/~conrad/249BW16Page/`, most proof will be omitted. My main motivation for learning this theory is that in the paper [1], they unified the Borel-Tits theory and Luna's theory of spherical systems.

2. Notation

In this note, I will fix k a characteristic zero field, G a connected reductive group over k, A a maximal k-split torus of G, P a minimal k-parabolic subgroup containing A.

3. Root systems

Let G be a connected semisimple groups over a field k, let A be a maximal split k-torus, we can define a set

$$\Phi_k := \Phi(G, A) = \{\text{non-trivial A-weights on } \mathfrak{g}\} \subset X^*(A) - \{0\}$$

since G is semisimple, we have Φ_k spans $X^*(A)_{\mathbb{Q}}$.

We have a bijection

$$\{\text{minimal parabolic k-subgroups contain A}\} \leftrightarrow \{\text{positive systems of roots } \Phi^+ \subset \Phi_k\} \tag{3.1}$$

given by $P \mapsto \Phi(P, A)$.

Theorem 3.1. *Let G be a connected reductive group over k and $A \subset G$ a maximal split torus. The map $P \mapsto \Phi_{P,k} := \Phi(P, A)$ is an inclusion-preserving bijection:*

$$\{\textit{parabolic k-subgroups contains } A\} \leftrightarrow \{\textit{parabolic subsets of } \Phi_k\} \tag{3.2}$$

Fixing P_0 a minimal k-parabolic subgroup, $\Phi_k^+ := \Phi(P_0, A)$ is a positive system of roots for Φ_k, let $\Delta_k \subset \Phi_k^+$ be the associated root basis, then we get an inclusion-preserving bijection

$$\{P_0 \subset Q\} \leftrightarrow \{\text{parabolic subsets } \Psi \subset \Phi_k \text{ containing } \Phi_k^+\} \tag{3.3}$$

Such Ψ are exactly the subsets $\Phi_k^+ \cup |I|$ for $I \subset \Delta_k$ where $[I] := (\mathbb{Z}I) \cap \Phi_k$. Informally, this tells us that standard parabolic subgroups are obtained from the minimal P_0 by admitting negative restricted roots supported in specific directions relative to restricted root basis attach to P_0.

3.1. Link between absolute and restricted roots.

Choose A a maximal split torus in G and a minimal parabolic k-subgroup P contains A. Pick a maximal k-torus T contains A and a Borel subgroup $T_{\overline{k}} \subset B \subset P_{\overline{k}}$. Inside the absolute root system $\Phi = \Phi(G_{\overline{k}}, T_{\overline{k}})$, we have a positive system of roots $\Phi^+ = \Phi(B, T_{\overline{k}})$. Let Δ be a basis of Φ^+, we are going to use Δ to construct a basis of $\Phi_k = \Phi(G, A)$ corresponding to its positive system of roots $\Phi(P, A) := \Phi_k^+$. Here note that we have chosen B inside $P_{\overline{k}}$.

Since $A_{\overline{k}} \subset T_{\overline{k}}$, we have a surjective restriction map

$$X^*(T_{\overline{k}}) \longrightarrow X^*(A_{\overline{k}})$$

sending Φ into $\Phi_k \cup \{0\}$, this carries Φ^+ into Φ_k^+ as we have chosen $B \subset P_{\overline{k}}$. Let $\Delta_0 = \{a \in \Delta : \ a|_{A_{\overline{k}}} = 1\}$. Let Δ_k be the image of $\Delta - \Delta_0$ in Φ_k, so $\Delta_k \subset \Phi_k^+$ since $B \subset P_{\overline{k}}$.

Lemma 3.2. *The parabolic subset $\Phi(P_{\overline{k}}, T_{\overline{k}}) \subset \Phi$ coincides with $\Phi^+ \cup [\Delta_0]$.*

Proof. Any parabolic subset containing Φ^+ has the form $\Phi^+ \cup [I]$ for a unique subset $I \subset \Delta$, $\Phi(P_{\overline{k}}, T_{\overline{k}})$ is such a subset since P is a parabolic and we have chosen $B \subset P_{\overline{k}}$.

I is the set of $a \in \Delta$ such that $-a \in \Phi(P_{\overline{k}}, T_{\overline{k}})$. Since $P = Z_G(A) \ltimes U$ with $U_{\overline{k}} \subset \mathscr{R}_u(B)$, since $-\Delta$ is disjoint from Φ^+, I is the set of $a \in \Delta$ such that

$$-a \in \Phi(Z_G(A)_{\overline{k}}, T_{\overline{k}}) = \{b \in \Phi : b|_{A_{\overline{k}}} = 1\}\}$$

which is to say $a \in \Delta_0$. □

Remark 3.3. We can use a refinement of the previous argument to show that Δ_0 is a basis of the root system $\Psi = \Phi(Z_G(A)_{\overline{k}}, T_{\overline{k}})$.

Proposition 3.4. *The set Δ_k defined above is the basis of Φ_k^+.*

Proof. We may assume that G is semisimple. Since $\Phi \subset \mathbb{Z}_{\geq 0}\Delta \cup \mathbb{Z}_{\leq 0}\Delta$, applying the restriction we get

$$\Phi_k \subset \mathbb{Z}_{\geq 0}(\Delta_k) \cup \mathbb{Z}_{\leq 0}(\Delta_k) \tag{3.4}$$

since we assume that G is semisimple, so we have Φ_k spans $X^*(A)_{\mathbb{Q}}$ and hence Δ_k spans $X^*(A)_{\mathbb{Q}}$.

It follows from (3.4) and the inclusion $\Delta_k \subset \Phi_k^+$ that Δ_k is a basis if it is linearly independent. Thus if we can prove $\#\Delta_k \leq \dim A$, then we will be done. This will be proved using the $*$-action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on the basis of $\Phi(B, T_{\overline{k}}) = \Phi^+ \subset \Phi$ 3.5. □

3.2. **Galois $*$-action.** We recall that there is a $*$-action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on Δ: for $\gamma \in \Gamma$, there exists a unique $\omega_\gamma \in W(\Phi) = N_G(T)(\overline{k})/T(\overline{k})$ such that $\omega_\gamma(\gamma(\Phi^+)) = \Phi^+$ and $\omega_\gamma(\gamma(\Delta)) = \Delta$. In general, ω_γ does not arise from $N_G(T)(k)$, and ω_γ must be nontrivial in the non-quasisplit case.

The $*$-action satisfies the following two properties:

(1) The restriction map $\mathrm{Res} : \Delta \to \Delta_k \cup \{0\}$ is Γ-invariant, so in particular of the fibers are Γ-stable.

(2) For a parabolic subgroup Q with $B \subset P_{\overline{k}} \subset Q$ corresponds to a unique subset of Δ containing Δ_0, and by lemma 3.2, this subset has the form $\Delta_0 \sqcup \Delta'$ for some $\Delta' \subset \Delta - \Delta_0$, we have the property that Q is defined over k if and only if $\Delta_0 \sqcup \Delta' \subset \Delta$ is Γ-stable or equivalently Δ' is Γ-stable.

Proposition 3.5. *Let G be a connected semisimple reductive group and A a maximal split torus, P a minimal parabolic k-subgroup contains A, and for T a k-maximal torus contains A, and a Borel subgroup $T_{\overline{k}} \subset B \subset P_{\overline{k}}$, Δ a basis of $\Phi^+ = \Phi(B, T_{\overline{k}})$ and $\Delta_k = Res'\Delta$, then we have $\dim A \geq \#\Delta_k$.*

Proof. Since $\dim A$ is the size of a basis of a restricted root system, by the correspondence between parabolic subgroups and the subsets of Δ_k, we have

$$2^{\dim A} \geq \#\{P \subset Q \text{ over k } \}$$

There is another way to describe the right hand side: it is the number of the Γ-stable subsets of $\Delta - \Delta_0$. But a Γ-stable subset is exactly a union of Γ-orbits, so the number of Γ-stable sets is $2^{\#\{\Gamma-\text{orbits}\}}$.

Now by (1) the number of Γ-orbits is at least the number of fibers of the Γ-invariant surjection $\Delta - \Delta_0 \to \Delta_k$, and the number of such fibers is $\#\Delta_k$, and we conclude that

$$2^{\dim A} \geq \#\{P \subset Q \text{ over k } \} = 2^{\#\{\Gamma-\text{orbits}\}} \geq 2^{\#\Delta_k}$$

so we get the inequality $\dim A \geq \#\Delta_k$. □

Corollary 3.6. *The fibers of $\Delta - \Delta_0 \to \Delta_k$ are the Γ-orbits away from Δ_0.*

4. Weyl group

Let G be a connected reductive group over a field k, A a maximal k-split torus in G, and P a minimal parabolic k-subgroup of G, let $N = N_G(A)$ and $Z = Z_G(A)$, so N/Z is a finite étale k-group. The group $N(k)/Z(k)$ is called the restricted Weyl group and we will denote it by W_k, we will show that it will be naturally identified with the combinatorial Weyl group $W(G, A)$ attached to the root system (G, A).

We want to study the relationship between $N(k)/Z(k)$ and N/Z, the first thing to notice is that

Lemma 4.1. *The finite étale k-group N/Z is constant. Equivalently the natural $Gal(\overline{k}/k)$-action on $(N/Z)(\overline{k}) = N(\overline{k})/Z(\overline{k})$ is trivial.*

Since the cosets of $Z_{\overline{k}}$ inside $N_{\overline{k}}$ are the connected components of $N_{\overline{k}}$, the triviality of the Galois action in the previous lemma implies that each components is defined over k. In other words, the connected components of N are *geometrically connected* over k. What is more subtle is that each of these components contain a k-point.

Proposition 4.2. *The natural map $N(k)/Z(k) \to (N/Z)(k)$ is surjective.*

Let W denote the constant finite k-scheme N/Z, the idea for proving the surjectivity is to show that $W(k)$ acts freely on a set whose $N(k)/Z(k)$-action is transitive. In the split case, there is a bijective correspondence between the set of Borel subgroups containing a given maximal torus and the set of Weyl chambers in the associated root system and there is a simply transitive action of the combinatorial Weyl group on the set of chambers, and there is also a transitive action of the Weyl group defined via the group data. In the general setting, we are led to consider the set $\mathscr{P}$ of minimal parabolic subgroups P of G that contain the maximal k-split torus A, there is a transitive $N(k)/Z(k)$ action on $\mathscr{P}$ as P, P' in $\mathscr{P}$ are $G(k)$-conjugate, there is also a free $W(k)$-action on $\mathscr{P}$ descend from $\overline{k}$.

Theorem 4.3. *The natural inclusion $W(\Phi_k) = W(\Phi(G, A)) \subset W(G, A)$ is an equality.*

Proof. The simply transitive property of $W(\Phi_k)$ on the positive system of roots $\subset \Phi_k$ is a general fact from the theory of root system as we have shown that Δ_k is the basis of Φ_k^+ 4.2. The group $W(G, A)$ acts simply transitively on the set $\mathscr{P}$ of minimal parabolic k-subgroups, and this follows from the relative Bruhat decomposition. By (3.1), there is a bijection between the set of positive system of roots and the minimal parabolic subgroups contain A, so we have $W(\Phi_k)$ and $W(G, A)$ act simply transitively on the same set $\mathscr{P}$, and hence they are equal. □

When G is quasisplit, i.e. the minimal parabolic k-subgroups are Borel subgroups, there is a refinement of the proposition 4.2 as follows: Let A be a maximal k-split torus in G and B a minimal parabolic k-subgroup containing A, so $Z_G(A)$ is a Levi subgroup of B, so it must be a k-maximal torus T. Any $g \in N_G(A)(k)$ normalizes T, so we have $N_G(A)(k) \subset N_G(T)(k)$, and this is an equality as any $g \in G(k)$ that normalizes T must also normalize the unique maximal k-split torus A in T. So by proposition 4.2, we have the inclusion of groups

$$W(\Phi(G, A)) = W(G, A)(k) = N_G(A)/Z_G(A)(k) = N_G(T)(k)/T(k) \subset (N_G(T)/T)(k) = W(G, T)(k)$$

Proposition 4.4. *Let G be a quasisplit connected reductive group over a field k, for a maximal split k-torus A and the associated maximal k-torus $T = Z_G(A)$. Let $W_k = N_G(A)(k)/Z_G(A)(k) = W(G, A)$ be the restricted Weyl group and $W = N_G(T)/T$ the absolute Weyl group, then the natural inclusion $W_k \hookrightarrow W(k)$ defined above is an equality.*

The equality $W_k = W(k)$ is clear when $H^1(k, T) = 1$, and this is the case when T is an induced torus. This is the case if the group G is semisimple of simple-connected or adjoint type. We can reduce the general quasisplit group case to the adjoint case as the restricted root system and the absolute root system are not changed under the the formation of central quotient.

5. Tits system

We say a k-semisimple group G k-isotropic if it contains $\mathbb{G}_m$ as a k-subgroup.

Theorem 5.1. *(Borel-Tits) Let G be a k-isotropic group, then all the maximal k-split torus A in G are $G(k)$-conjugate, the set $\Phi(G, A)$ of nontrivial A-weights on Lie(G) is a root system in $X^*(A)_{\mathbb{Q}}$ and the minimal parabolic k-subgroups P in G are $G(k)$-conjugate.*

Every P contains some A, and every A lies in some P, the assignment $P \mapsto \Phi(P, A)$ is a bijection from the set of minimal parabolic k-subgroups P contains A onto the set of positive systems of roots in $\Phi(G, A)$.

The étale k-group $W(G, A) := N_G(A)/Z_G(A)$ is constant, and $N_G(A)(k)/Z_G(A)(k) \to W(G, A)(k)$ is an isomorphism, and naturally $W(G, A)(k) = W(\Phi(G, A))$.

The common dimension of maximal k-split tori in G is called the k-rank. The isomorphism

$$N_G(A)/Z_G(A)(k) \cong W(G, A)(k)$$

is quite remarkable as in general $H^1(k, Z_G(A)) \neq 1$, so it doesn't follow directly from the cohomological argument.

A big challenge in proving these results is that *we cannot use extension of the ground field easily as in the split case.*

Definition 5.2. A *Tits system* is a 4-tuple $(\mathscr{G}, B, N, \Sigma)$ where $\mathscr{G}$ is an abstract group, B and N are subgroups, and $\Sigma \subseteq N/(B \cap N)$ is a subset such that the following axioms are satisifed:

(T1) $B \cup N$ generates $\mathscr{G}$ and $B \cap N$ is normal in N.
(T2) the elements of Σ have order 2 in the quotient $W := N/(B \cap N)$ and generate W.
(T3) for all $\sigma \in \Sigma$ and $\omega \in W$, $\sigma B \omega \subseteq B\omega B \cup B\sigma\omega B$ (using any representatives for σ and ω in N).
(T4) $\sigma B \sigma \nsubseteq B$ for all $\sigma \in \Sigma$.

Let's remark that Σ is uniquely determined by $(\mathscr{G}, B, N)$.

Theorem 5.3. *(Borel-Tits) Let $N = N_G(A)$ and $Z = Z_G(A)$, and P a minimal k-parabolic subgroup of G containing A. Let Δ_k be the basis of the positive system of roots $\Phi_k^+ = \Phi(P, A)$, and let $R = \{r_a | \ a \in \Delta_k\}$ be the associated set of simple positive reflections. The 4-tuple $(G(k), P(k), N(k), R)$ is a Tits system with Weyl group $W_k = N(k)/Z(k)$.*

This is the *standard Tits system* associated to (G, A, P), since $P(k) \cap N(k) = Z(k)$, we have $N(k)/(P(k) \cap N(k)) =: W_k = W(\Phi_k)$, that is the Weyl group of the standard Tits system coincides with $W(G, S)$.

The axiom $(T1)$ follows from the relative Bruhat decomposition $G(k) = \sqcup_{\omega \in W_k} P(k) n_\omega P(k)$. The quotient group $N(k)/Z(k) = W_k$ is generated by R, and this is the axiom $(T2)$.

To prove $(T4)$, it suffices to prove that $rP(k)r \neq P(k)$ for $r := r_a \in R$ with $a \in \Delta_k$. Since rPr contains $rU_a r = U_{r(a)} = U_{-a}$, it suffices to prove $U_{-a}(k)$ is not contained in $P(k)$. Since for any closed smooth connected k-subgroup H of G, we have $U_{-a} \cap H$ is also smooth connected, so $U_{-a} \cap P$ is smooth and connected, but its Lie algebra is trivial, so $U_{-a} \cap P = 1$, and hence $U_{-a}(k) \cap P(k) = 1$, which imply that $rP(k)r \neq P(k)$ as $U_{-a}(k) \neq 1$ as the smooth connected k-group U_{-a} is nontrivial and k-split.

Axiom $(T3)$ follows from the following inclusion: for $a \in \Delta_k$ and $\omega \in W_k$, we have:

$$rP(k)\{\omega, r\omega\}P(k) \subseteq P(k)\{\omega, r\omega\}P(k)$$

References

[1] Friedrich Knop and Bernhard Krötz. Reductive group actions. *arXiv preprint arXiv:1604.01005*, 2016.