BASE CHANGE AND THE ADVANCED THEORY OF THE TRACE FORMULA

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1. Introduction

This is a study note for the book.

2. Introduction from the book

The general theory of automorphic forms is in some ways still young, it is expected to play a fundamental unifying role in a wide array of arithmetic questions. Much of this can be summarized as Langlands' functoriality principle. For two reductive groups G and G' over a number field F, and a map ${}^LG' \to {}^LG$ between their L-groups, there should be an associated correspondence between their automorphic representations.

There is an important special case where the functoriality is more accessible, it is roughly speaking, the case that ${}^LG'$ is the group of fixed points of an automorphism of ${}^LG$. Assume that G' is quasi-split, then G' is called a twisted endoscopic group for G. Endoscopy groups were introduced by Langlands and Shelstad to deal with the problems that arise originally in connection with Shimura varieties. Besides being a substantial case of the general question, a proper understanding of functoriality for endoscopic groups would be significant in its own right. It would impose an internal structure on the automorphic representations of G, namely a partition into "L-packets", which would be a prerequisite to understand the nature of the general functoriality correspondence. However the problem of functoriality for endoscopic groups appears accessible only in comparison with the general case.

When the endoscopic group G' equals to $GL(2)$, Jacquet and Langlands solved the problem using the trace formula for $GL(2)$. In general, it will be necessary to deal simultaneously with a number of endoscopic groups G', namely the ones associated to those automorphisms of ${}^LG^0$ which differ by an inner autormophism. One would hope to compare a twisted trace formula for G with some combination of trace formulas for the relevant groups G'. There now exists a trace formula for general groups. The purpose of this book is to test these methods on the simplest case of general rank. We will assume that G' is equal to $GL(n)$. A special feature of this case is that there is essentially only one endoscopic group to be considered.

We will consider the case where the identity component of the L-group G^0 is isomorphic to ℓ-copies of $GL(n, \mathbb{C})$ and G' comes from the diagonal image of $GL(n, \mathbb{C})$, the fixed point set of the permutation automrophism. This is the problem of cyclic base change for $GL(n)$. We will compare the trace formula of G with that of G'. For each term in the trace formula of G, we will construct a companion term from the trace formula of G'. One of the main results is that these two sets of terms are equal. More or less, this means that there is a term by term identification of the trace formulas for G and G'.

A key constituent in the trace formula of G comes from the right convolution of a function $f \in C_c^\infty(G(\mathbb{A}))$ on the subspace of $L^2(G^0(F)\backslash G^0(\mathbb{A})^1)$ which decomposes discretely. However this is only one of several such collection of terms, which are parametrized by the Levi components M in G. Together, they form a "discrete part" of the trace formula

$$I_{disc,t}(f) = \sum_M ||W_0^M||\ ||W_0^G||^{-1} \sum_{s \in W(\mathfrak{a}_M)_{\text{reg}}} ||\det(s-1)_{\mathfrak{a}_M^G}||^{-1}\ \text{tr}(M(s,0)\rho_{P,t}(0,f))$$

in which $\rho_{P,t}$ is a representation induced from the discrete spectrum of M and $M(s,0)$ is an intertwining operator. Let S be a finite set of valuations of F which contains all the Archimedean and ramified places. For each $v \in S$, let f_v be a fixed function in $C_c^\infty(G(F_v))$. We then define a variable function

$$f = \prod_v f_v$$

Date: September 2025.

in $C_c^\infty(G(\mathbb{A}))$ by choosing functions $\{f_v \ v \notin S\}$ which are spherical bi-invariant under the maximal compact subgroup of $G^0(F_v)$. For each valuation v not in S, the Satake transform provides a canonical map $f_v \to f'_v$ from the spherical functions on $G(F_v)$ to the spherical functions on $G'(F_v)$. Our result imply that there are fixed functions $f'_v \in C_c^\infty(G'(F_v))$ for the valuations v in S with the property that if

$$f' = \prod_v f'_v$$

then

$$I^G_{disc,t}(f) = I^{G'}_{disc,t}(f') \tag{2.1}$$

this identity will impose a strong relation between the autormophic representations of G and G'. In particular, we will use it to establish the global base change for $GL(n)$.

Chapter 1 is devoted to the correspondence $f_v \leftrightarrow f'_v$. We shall also describe a dual correspondence between tempered representations of $G(F_v)$ and $G'(F_v)$. The correspondence is defined by comparing orbital integrals, for a given f_v, we shall sow that there exists a function $f'_v \in C_c^\infty(G'(F_v))$ whose orbital integrals match those of f_v under the image of the norm map from $G(F_v)$ to $G'(F_v)$. At the p-adic cases we shall do this by an argument of descent. The main new aspect of Chapter 1 is the proof that the matching of orbital integrals is compatible with the canonical map of spherical functions.

In Chapter 2 we shall compare two trace formulas. Theorem A establishes an identification of the geometric terms on the left-hand sides of the two formulas, while theorem B gives parallel identities for the spectral terms on the right. The two theorems will be proved together by means of an induction argument.

As an application of the identity 2.1, we shall establish base change for $GL(n)$ in chapter 3. For $GL(2)$, the complete spectral decomposition of the space of autormophic forms is known, and this makes it possible to compare very explicitly the discrete spectra of $GL(2,\mathbb{A}_F)$ and $GL(2,\mathbb{A}_E)$. Such explicit information is not available for $n > 2$.

Assume that E/F is a cyclic extension of number fields, of prime degree ℓ with Galois group

$$\{1, \sigma, \sigma^2, \ \cdots, \sigma^{\ell-1}\}$$

given the local lifting, we may define the global lifting as follows: let $\pi = \otimes_v \pi_v$ be an automorphic representation of $GL(n,\mathbb{A}_F)$, a tensor product over all places v of F, let $\Pi = \otimes_\omega \Pi_\omega$ be an automorphic representation of $GL(n,\mathbb{A}_E)$, ω denoting a place of E. We say that is a strong base change lift of π if for any $\omega|v$, Π_ω lifts π_v. Our main theorem is theorem 3.5.2 and applies to representations *induced from cuspidal*. Let π, Π stand for such representations of $GL(n,\mathbb{A}_F)$, $GL(n,\mathbb{A}_E)$. We prove that

- If Π is σ-stable, i.e. Π is equivalent to $\Pi \circ \sigma$- is is base change lift of finitely many π.
- Conversely, given π, there is a unique σ-stable Π lifting π.

Assume that Π is a *cuspidal* representation of $GL(n,\mathbb{A}_E)$. We show that

- If $\Pi \cong \Pi^\sigma$, there are exactly ℓ representation π lifted by Π, they are all twists of one of them by powers of the class field character associated to E/F.
- Assume Π is not isomorphic to Π^σ, then the data $(\Pi, \Pi^\sigma, \cdots, \Pi^{\sigma^{\ell-1}})$ define through the theory of Eisenstein series, an automorphic representation of $GL(n\ell,\mathbb{A}_E)$. This representation is σ-stable and lifts exactly one cuspidal representation π of $GL(n\ell,\mathbb{A}_F)$.

the last two parts will imply the existence of an automrophic induction functor sending automorphic representations of $GL(n,\mathbb{A}_E)$ to those of $GL(n\ell,\mathbb{A}_E)$.

3. Base change

3.1. **Weak and strong base change: definitions.** In this section E/F will denote a cyclic extension of degree ℓ of number fields, we write v for the places of F, ω for the places of E, in particular $\mathbb{A} = \mathbb{A}_F$, $\mathbb{A}_E = \mathbb{A} \otimes E$, $G = GL(n)$. If π is an automorphic representation of $G(\mathbb{A})$, we have $\pi = \otimes_v \pi_v$ where π_v is unramified for almost all v.

For any finite prime v unramified in E, we have the base change homomorphism $b: \ \mathcal{H}_{E_v} \to \mathcal{H}_{F_v}$, by duality, to an unramified representation π_v, we may associated an unramified representation $\Pi_v = \otimes_{\omega|v} \Pi_\omega$ of $G(E_v) = \prod_{\omega|v} \ G(E_\omega)$.

In terms of the Hecke eigenvalues, the correspondence is described as follows, if f_v is the residue degree of E above an unramified v, then for any $\omega|v$

$$(t_{\pi_v})^{f_v} = t_{\Pi,\omega} \tag{3.1}$$

Definition 3.1. Let $\pi,\ \Pi$ denote autormophic reoresentations of $G(\mathbb{A})$, $G(\mathbb{A}_E)$, we say that Π is a weak base change lift of π if the relation (3.1) is satisfied for almost all finite places $v,\ \omega$.

This definite may be strengthened using the theory of local Base change

Definition 3.2. We say that Π is a strong base change lift of π if for any $\omega|v$, the component Π_ω is a base change lift of π_v.

we will use this definition when the compoenents of π and Π are generic, and hence base change is expressed by character identities.

3.2. Fibers of global base change.

Theorem 3.3. *Let $\pi,\ \pi'$ be cuspidal autormophic representations of $G(\mathbb{A})$, assume that for almost v*

$$(t_{\pi,v})^{f_v} = (t_{\pi',v})^{f_v}$$

then $\pi = \pi' \otimes \chi$ for some character χ of $F^\times N(\mathbb{A}_E^*)\backslash\mathbb{A}^*$.

Proof. Let η be a character of $\mathbb{A}^*$ vanishing exactly on $F^*N(\mathbb{A}_E^*)$, we compare the products $\prod_{i=1}^{\ell} L^S(s,\pi\otimes\tilde{\pi}'\otimes\eta^i)$ and $\prod_{i=1}^{\ell} L^S(s,\pi\otimes\tilde{\pi}\otimes\eta^i)$. where $\tilde{\sigma}$ denotes the contragredient of σ.

If v is a finite place of F, the factor of the first product at v is equal to the inverse of

$$\prod_{i=1}^{\ell} \det(1 - t_v \otimes \tilde{t_v}'\zeta_v^i q_v^{-s})$$

with $\tilde{t_v}'$ denote the adjoint of t_v' and $\zeta_v = \eta(\tilde{\omega}_v)$. We take S large enough so that E/F is unramified for $v \notin S$, then ζ_v is a root of unity of order f_v. Consequently, this product is equal to

$$\det(1 - t_v^{f_v} \otimes (t_v')^{f_v} q_v^{-f_v s})^{\ell/f_v}$$

which by assumption is equal to

$$\prod_{i=1}^{\ell} L^S(s,\pi\otimes\tilde{\pi}\otimes\eta^i) = \prod_{i=1}^{\ell} L^S(s,\pi\otimes\tilde{\pi}\otimes\eta^i)$$

we may assume $\pi,\ \pi'$ unitary, the product on the right has a pole at $s=1$, hence also the left one, and since the terms doesn't vanish on $\mathrm{Re}(s)=1$, from a result of Jacquet and Shalika we see that $\pi' = \pi\otimes\eta^i$ for some i. ☐

3.3. Weak lifting.

Definition 3.4. We will say that the automorphic representation π of $G(\mathbb{A})$ is induced from cuspidal if there is a cuspidal unitary representation σ of $M(\mathbb{A})$ where $P = MN$ is an F-parabolic subgroup of G such that

$$\pi = \mathrm{ind}_{M(\mathbb{A})N(\mathbb{A})}^{G(\mathbb{A})}\ (\sigma\otimes 1)$$

We now state the main result on the base change concerning the base change for cyclic representations of *prime degree*. We will say that Π lifts π if Π is a weak lifting of π 3.1.

Theorem 3.5. *All representations are induced from cuspidal, E/F is cyclic of prime degree ℓ.*

- *Assume that π is cuspidal, π is not isomorphic to $\pi\otimes\eta$ for any η, then there is a unique σ-stable representation Π of $G(\mathbb{A}_E)$ lifting π and Π is cuspidal.*
- *Assume $\pi \cong \pi\otimes\eta$ for some η. Then there is a cuspidal representation Π_1 of $GL(n/\ell,\mathbb{A}_E)$ with Π_1 not isomorphic to Π_1^σ, such that $\Pi = \Pi_1\times\cdots\times\Pi^{\sigma^{\ell-1}}$ is the only lift of π.*

3.4. **Strong lifting.**

Theorem 3.6. *(Strong lifting) Assume that* π, Π *are representations induced from cupidal of* $G(\mathbb{A})$, $G(\mathbb{A}_E)$ *respectively.*

If Π *is a weak lifting of* π*, then* Π *is in fact a strong lifting of* π*.*