AUTOMORPHISMS OF MULTIPLICITY FREE HAMILTONIAN MANIFOLDS

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1. Introduction

This is my study note for Knop's paper [Kno11]. Consider a connected compact Lie group K acting on a connected Hamiltonian manifold M, a measure of the complexity of M is half of the dimension of the symplectic reductions of M, and it is natural to study Hamiltonian manifolds with low complexity, starting with the case of complexity zero, the so-called *multiplicity free* manifolds. It has been a long standing problem to complete this project, Delzant conjectured in 1989 that any compact multiplicity free space is uniquely determined by two invariants: its momentum polytope $\mathcal{P}$ and its principal isotropy group L_0. Evidence for this conjecture was Delzant's celebrated classification of multiplicity free torus actions.

One knows that a multiplicity free manifold is characterized by the combinatorial data $(\mathcal{P}, L_0)$, it is natural to ask which pairs arise in this way and we show that this can be reduced to a purely local problem on $\mathcal{P}$. More precisely, $\mathcal{P}$ has to look locally like the weight monoid of a smooth affine spherical variety, see theorem 5.2.

Let's describe briefly the local problem, let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subsalgebra, then it is well known that the orbit space $\mathfrak{k}^*/K$ can be identified with a Weyl chamber $\mathfrak{t}^+ \subseteq \mathfrak{t}$, thus the moment map $m: M \to \mathfrak{k}^*$ gives rise to the invariant moment map $\psi: M \to \mathfrak{t}^+$. By a celebrated theorem of Kirwan, the image $\mathcal{P} = \psi(M)$ is a convex polytope for M compact. The local statement asserts that two compact multiplicity free manifolds with the same moment polytope $\mathcal{P}$ and the same pricipal isotropy group are isomorphic locally over $\mathcal{P}$ (see theorem 2.5). Using some techniques, one can reduce this local problem to a statement about smooth affine spherical varieties, the "Knop conjecture", see theorem 2.7.

To pass from local to global, we need to determine the automorphism group of a multiplicity free manifold, more precisely, we need to know the *sheaf of automorphisms* $\mathcal{A}_M$ of M over $\mathcal{P}$. The main result is

Theorem 1.1. *Let M be a connected multiplicity free manifold with moment polytope $\mathcal{P}$, then $\mathcal{A}_M$ is a sheaf of abelian groups, moreover, all of its higher cohomology groups vanish: $H^i(\mathcal{P}, \mathcal{A}_M) = 0$ for $i \geq 1$.*

As an application, the vanishing of H^1 implies that two compact multiplicity free manifolds which are locally isomorphic over $\mathcal{P}$ are isomorphic globally. Together with the local statement this gives the Delzant conjecture. The vanishing of H^2 implies that there are no obstructions for gluing manifolds which are given locally over $\mathcal{P}$ to one global manifold M.

The bulk of Knop's paper is to show that the sheaf $\mathcal{A}_M$ is controlled by certain root system Φ_M. In some sense, this root system is a symplectic analog of the restricted root system of a symmetric space and is fundamental in understanding the geometry of a multiplicity free manifold.

To determine all automorphisms we have to study K-invariant smooth functions on M since these generate automorphisms via Hamiltonian flows. It will turn out that (see theorem 4.1) the smooth K-invariants are controlled by a finite reflection group W_M, this group is then used to construct the root system Φ_M.

We conclude this introduction with some historical remarks: The local-to-global principle, theorem 1.1 was formulated by Knop first, and he formulated the local statement as a conjecture (theorem 2.5 and 2.7). Meanwhile, Luna launched a program to classify all spherical varieties and completed it for groups of type A, this enabled his student Camus to settle the Knop conjecture for groups of type A, Losev managed to bypass all problems which still exist in Luna's program and proved the conjecture in full generality. This enabled Knop to finally complete the proof of the Delzant conjecture.

1.1. **Notation.** In the following, K will be a compact connnected Lie group with Lie algebra $\mathfrak{k}$, $T_{\mathbb{R}} \subseteq K$ will be a maximal torus with Lie algebra $\mathfrak{t}_{\mathbb{R}}$ and Weyl group W, we fix a Weyl chamber $\mathfrak{t}^+ \subseteq \mathfrak{t}^*_{\mathbb{R}}$.

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2. The local Delzant conjecture

Let M be a connected Hamiltonian K-manifold with moment map $m : M \to \mathfrak{k}^*$, the moment image of M is the set $\mathcal{P} = m(M) \cap \mathfrak{t}^+$, a theorem of Kirwan states that $\mathcal{P}$ is a convex polyhedron when M is compact. We define the invariant moment map as the composed map

$$\psi : M \to \mathfrak{k}^* \to \mathfrak{k}^*/K \to \mathfrak{t}^+$$

the polytope is the image of ψ.

Definition 2.1. A Hamiltonian K-manifold M is *multiplicity free* if it is connected and if $\dim M/K = \dim \mathcal{P}$.

Definition 2.2. A multiplicity free manifold is convex if

- the momentum image $\mathcal{P}$ is convex.
- the invariant moment map $\psi : M \to \mathcal{P}$ is proper.

For the remainder of this section, we will assume that M is convex and multiplicity free.

We recall some facts about the principal isotropy group: let $\mathfrak{a}^0 \subseteq \mathfrak{t}^*_{\mathbb{R}}$ be the affine subspace spanned by $\mathcal{P}$, the interior of $\mathcal{P}$ inside $\mathfrak{a}^0$ is called its relative interior, it is open and dense in $\mathcal{P}$. The centralizer $L_{\mathbb{R}}$ of $\mathfrak{a}^0$ is a Levi subgroup of K containing the maximal torus T, the generic structure of M is then described by the following well-known theorem

Lemma 2.3. *Let $\Sigma := m^{-1}(\mathcal{P}^0)$, then:*

- *Σ is a Hamiltonian $L_{\mathbb{R}}$-manifold with moment map $m|\Sigma$.*
- *Let L_0 be the kernel of $L_{\mathbb{R}}$ action on Σ, then $A_{\mathbb{R}} = L_{\mathbb{R}}/L_0$ is a torus acting freely on Σ.*
- *The map $K \times^{L_{\mathbb{R}}} \Sigma$ is an open immersion, hence L_0 is a principal isotropy group for the K-action.*
- *$M/K\Sigma$ has codimension ≥ 2.*

The proof is reduced to the toric variety case.

Remark 2.4. It is possible to show that $K\Sigma$ is exactly the union of all K-orbits of maximal dimension.

The moment image $\mathcal{P}$ also determines the Lie algebras $\mathfrak{l}_{\mathbb{R}}$ and $\mathfrak{l}_0$ of $L_{\mathbb{R}}$ and L_0, $\mathfrak{l}_{\mathbb{R}}$ is the centralizer of $\mathcal{P}$ and $\mathfrak{l}_0$ is the set of ξ with $\chi_1(\xi) = \chi_2(\xi)$ for all $\chi_1,\ \chi_2 \in \mathcal{P}$.

Theorem 2.5. *Let $M_1,\ M_2$ be convex multiplicity free Hamiltonian K-manifolds with invariant moment map $\psi_1,\ \psi_2$, assume $\psi_1(M_1) = \psi_2(M_2) = \mathcal{P}$, and $L_0(M_1) = L_0(M_2)$, then every $a \in \mathcal{P}$ has a convex open neighborhood U such that $(M_1)_U \cong (M_2)_U$ as a Hamiltonian K-manifold.*

Let M be any convex Hamiltonian manifold and let $x \in M$ be a point with $m(x) = 0$, then the symplectic slice theorem asserts that a neighborhood of Kx in M is uniquely determined by two data: the isotropy group $H_{\mathbb{R}} = K_x$ and the symplectic slice $S = (\mathfrak{k}x)^{\perp}/(\mathfrak{k}x)$. Now put $X = G \times^H S$.

Theorem 2.6. *Let $X = G \times^H S$ as above, then*

- *X is a multiplicity free K-manifold if and only if X is a spherical G-variety.*
- *Let $\mathcal{P}_X$ be the moment image and of X and $\mathcal{Q}$ the convex cone generated by the weight monoid of X, then $\mathcal{P} = i\mathcal{Q}$.*
- *Let Λ_X be the lattice determining the principal isotropy group L_0 of X, then $\Lambda_X = \langle \Xi_X \rangle$.*

Now the local Delzant conjecture following from the following theorem

Theorem 2.7. *Two smooth affine spherical G-varieties with the same weight monoid are G-equivariantly isomorphic.*

3. The local model and its invariants

Let M be a Hamiltonian manifold and $x \in M$ a point with $m(x) = 0$, then the slice theorem asserts that a neighborhood of Kx is determined by two data: the isotropy group $H_{\mathbb{R}} = K_x$ and the space $S = (\mathfrak{k}x)^{\perp}/(\mathfrak{k}x)$, considered as a symplectic representation of $H_{\mathbb{R}}$, let $G,\ H$ be the complexifications of K and $H_{\mathbb{R}}$, we worked with the local model $X = G \times^H S$, and we consider

$$\mathscr{M} := K \times^{H_{\mathbb{R}}} (\mathfrak{h}_{\mathbb{R}}^{\perp} \oplus S)$$

let $\mathscr{M}^c$ be the complexification of $\mathscr{M}$, then we have $\mathscr{M}^c$ is isomorphic to the cotangent bundle T^*X of X as a Hamiltonian G-variety, $\mathscr{M}$ is multiplicity free if and only if X is spherical.

The G-invariants of T^*X has been determined in [Kno90]

Theorem 3.1. *Let X be a smooth affine spherical variety, let $\Lambda_X \subseteq \mathfrak{t}^*$ be the subgroup generated by the weight monoid and let $\mathfrak{a}^*$ be its $\mathbb{C}$-span, then there is a finite subgroup $W_X \subseteq GL(\mathfrak{a}^*)$ and a morphism $q: T^*X \to \mathfrak{a}^*/W_X$ with*

- *q is the categorical quotient of T^*X by G.*
- *W_X is generated by reflections.*
- *W_X is a subgroup of $N_W(\mathfrak{a}^*)/C_W(\mathfrak{a}^*)$ and normalizes the lattice $\Lambda_X \subseteq \mathfrak{a}^*$.*

Theorem 3.2. *Assume that $\mathscr{M}$ is multiplicity free, then there is $n \in N_W(\mathfrak{a}^*)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & \mathscr{M}^c \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathfrak{a}^*_{\mathbb{R}} & \longrightarrow & \mathfrak{a}^*/W_{\mathscr{M}} \end{array}$$

here $W_{\mathscr{M}} := {}^nW_X$.

Now we can determine the smooth K-invariants on $\mathscr{M}$

Corollary 3.3. *The composed map $\psi/W_{\mathscr{M}}: \mathscr{M} \to \mathfrak{a}^*_{\mathbb{R}} \to \mathfrak{a}^*_{\mathbb{R}}/W_{\mathscr{M}}$ is smooth and*

$$(\psi/W_{\mathscr{M}})^*: C^\infty(\mathfrak{a}^*_{\mathbb{R}})^{W_{\mathscr{M}}} \longrightarrow C^\infty(\mathscr{M})^K$$

is surjective.

4. Invariants of multiplicity free manifolds

In this section, we will determine the smooth invariants on an arbitrary convex multiplicity free manifolds.

Theorem 4.1. *Let M be a convex multiplicity free Hamiltonian manifold with momentum image $\mathcal{P} = \psi(M)$, let $\mathfrak{a}^0 \subseteq \mathfrak{t}^*_{\mathbb{R}}$ be the affine space spanned by $\mathcal{P}$, then*

- *There is a finite group $W_0 \subseteq N_W(\mathfrak{a}^0)/C_W(\mathfrak{a}^0)$ such that the composed map $\psi/W_0: M \to \mathfrak{a}^0/W_0$ is smooth and the induced map*
$$(\psi/W_0)^*: C^\infty(\mathcal{P}/W_0) \cong C^\infty(M)^K$$
- *Among all groups W_0, there is a unique minimal one, denoted by W_M, it is characterized by the fact that it is generated by reflections which have a fixed point in $\mathcal{P}$.*

We first have a local version of this theorem

Lemma 4.2. *For every $a \in \mathcal{P}$ there is a unqiue subgroup $W(a)$ of $N_W(\mathfrak{a}^0)/C_W(\mathfrak{a}^0)$ having a as a fixed point and an open convex neighborhood U of a in $\mathcal{P}$ such that*

$$C^\infty(U/W(a)) \cong C^\infty(M_U)^K$$

where $M_U = \psi^{-1}(U)$ and $W(a)$ is generated by reflections.

and we can show that all the local Weyl groups $W(a)$ can be glued together.

5. The classification of multiplicity free manifolds

We are able to complete the classification of convex multiplicity free manifolds. For this, let $\Phi = (\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$ be a root datum, choosing a system Δ^+ of positive roots, these data define the usual tuple $T \subseteq B \subseteq G$, where G is a connected complex reductive group, B is a Borel subgroup, T is a maximal torus.

Definition 5.1. Let $\Lambda_0 \subseteq \Lambda$ be a subgroup, then a cone $\mathcal{Q} \subseteq \Lambda \otimes \mathbb{R}$ is called multiplicity free for $(\Phi, \Delta^+, \Lambda_0)$ if there is a smooth affine spherical G-variety X with weight lattice Ξ_X satisfies

- $\mathcal{Q}$ the convex cone generated by Ξ_X.

- Λ_0 the abelian group generated by Ξ_X.

Theorem 5.2. *Let K be a compact connected Lie group with root datum Φ, and a choice of positive roots Δ^+, then there is a bijection between the isomorphism classes of convex multiplicity free K-manifolds and pairs $(\mathcal{Q}, \Lambda_0)$ such that*

- *Λ_0 is a subgroup of Λ and $\mathcal{Q}$ is a locally polyhedral convex subset of the Weyl chamber determined by Δ^+.*
- *the tangent cone of every $a \in \mathcal{Q}$ is multiplicity free for the tuple $(\Phi_a, \Delta_a^+, \Lambda_0)$.*

We can recover the Delzant's theorem.

Theorem 5.3. *Let $K = T_{\mathbb{R}}$ be a torus, then the compact multiplicity free $T_{\mathbb{R}}$-manifolds with $T_{\mathbb{R}}$ acting effectively are classified by compact simple regular polytopes $\mathcal{P}$.*

References

[Kno90] Friedrich Knop. Weylgruppe und momentabbildung. *Inventiones mathematicae*, 99(1):1–23, 1990.

[Kno11] Friedrich Knop. Automorphisms of multiplicity free hamiltonian manifolds. *Journal of the American Mathematical Society*, 24(2):567–601, 2011.