AUTOMORPHISM GROUPS OF SPHERICAL VARIETY

Contents

1. Introduction

This is my study note for the various result on automorphism groups of spherical varieties, they are related to different subsets of spherical roots.

2. Notation

For the purpose of future application to non algebraically closed field, we will keep the notation compatible with [KK16], for the application to harmonic analysis, we change some notations to be compatible with [SV17].

We will let k be a characteristic zero field, and G a connected reductive group over k.

X a spherical variety for G, and we introduce the following invariants for X: A_X the maximal torus of X which is a quotient of A, $\chi(X)$ the characters of B-semiinvariant functions on X.

3. Result of Knop

In this section, we will assume k is algebraically closed, we will summarize the main result of the paper [Kno96]. We will denote $\Lambda(X) = \chi(X)^*$, the cocharacter lattice of X, $\mathfrak{a}_X = \Lambda(X) \otimes \mathbb{Q}$. An B-invariant $\mathbb{Q}$-valued valuation on $k(X)$ which is trivial on $k^\times$ will induce an element of $\Lambda(X)$ via restriction to $k(X)^{(B)}$, here we use the identification $\chi(X) \cong X^*(A_X)$. We will denote $\mathscr{V} \subset \mathfrak{a}_X$ the cone generated by the images of G-invariant valuations, $\mathscr{V}$ contains the image of negative Weyl chamber under the natural map $\mathfrak{a} \to \mathfrak{a}_X$.

The cone $\mathscr{V}$ is the fundamental domain for a finite reflection group $W_X \subset \mathrm{End}(\mathfrak{a}_X)$. We also consider the cone dual to $\mathscr{V}$

$$\mathscr{V}^\perp = \{\chi \otimes \chi(X) \otimes \mathbb{R} | \ \langle \chi, v\rangle \leq 0 \text{ for every } v \in \mathscr{V}\}$$

The generators of the intersections of its external rays with $\chi(X)$ is *spherical roots of* X, which we denote by Σ_X.

Remark 3.1. This is the so called *primitive spherical roots* in the literature.

Remark 3.2. We will see later that a set of roots Σ_X^n related to Σ_X, are simple roots of a based root system with Weyl group W_X, this root system is the *spherical root system* of X.

We will introduce the following types of spherical roots, they will play an important role in the calculation of various automorphism groups and in fact also in the proof of 4.4.

Definition 3.3. We define the following for types of spherical roots, $\Sigma_X^1, \Sigma_X^2, \Sigma_X^3, \Sigma_X^4$

(1) $\gamma \in \Sigma_G$ and there exists $D \in \mathscr{D}$ with $\rho(D) = \frac{1}{2}\alpha^\vee|_{\mathfrak{a}_X}$.

(2) there is a subset $\Sigma \subset \Sigma_G$ of type $B_n,\ n \geq 2$ such that

$$\gamma = \alpha_1 + \alpha_2 + \cdots \alpha_n$$

and $\alpha_i \in \Sigma_X^p$ for $i > 1$.
(3) there is a subset $\{\alpha_1,\ \alpha_2\} \subset \Sigma_G$ of type G_2 with α_1 short root and $\gamma = 2\alpha_1 + \alpha_2$.
(4) $\gamma \in X^*(A_X)$ but $\gamma \notin \mathbb{Z}\Phi$.

3.1. **Group schemes.** In this section, we will recall Knop's theory on the construction of the root system using the geometry of the moment map and integration of the Lie algebra action. k will be an algebraically closed field of characteristic zero, G will be a connected reductive group over k and B the Borel subgroup of G, T a maximal torus of B.

Let $S' \to S$ be a morphism of varieties, Z'/S' is an S'-scheme, we will denote the Weil restriction of Z' along φ as $\mathrm{Res}_{S'/S}Z'$, when Z'/S' is an S'-group scheme, we can show $\prod_{S'/S} Z'$ is an S-group scheme.

Example 3.4. Let ℓ/k be a field extension, when $\varphi : \mathrm{Spec}(\ell) \to \mathrm{Spec}(k)$ is induced by field inclusion, and G_ℓ is an algebraic group scheme over ℓ, then the restriction of scalar coincide with the usual restriction of scalars for algebraic groups, $\mathrm{Res}_{\ell/k}G_\ell$ is an algebraic group scheme over k.

We now apply the restriction of scalar operation to the following situation: Let S' be an affine space and W a finite group acting linearly on S', we assume W is generated by reflections. Then $S = S'/W$ is also an affine space and S'/S is finite and flat. Assume W also acts on a finitely generated free abelian group Γ, let A be the algebraic torus with character lattice Γ, then we know that

$$Z := \prod_{S'/S} (A \times S')$$

exists and is a smooth, commutative S-group scheme.

Let X/S be an S-scheme, W acts on $\mathrm{Mor}_S(X, Z)$ and hence on the S-group scheme Z, we define

$$\mathcal{A} = \mathcal{A}(W, S', \Gamma) := Z^W$$

as the set of fixed points of W-action.

Lemma 3.5. *$\mathcal{A}/S$ is a smooth commutative affine group scheme.*

Next we investigate the fibers $\mathcal{A}_s := \pi^{-1}(s) \subseteq \mathcal{A}$, it is an affine commutative group, it decomposes into unipotent and semisimple part $\mathcal{A}_s = \mathcal{A}_s^u \times \mathcal{A}_s^s$.

Lemma 3.6. *Let $s \in S$, then for every $s' \in S'$ there is a homomorphism $\iota_{s'} : \mathcal{A}_s \to \mathcal{A}$ with kernel $\mathcal{A}_s^u$ and image $A^{W_{s'}} \cong \mathcal{A}_s^s$.*

Lemma 3.7. *The set of global sections of $\mathcal{A}/S$ equals to A^W, and $\sigma(s) \in \mathcal{A}_s$ is semisimple for every $s \in S$.*

We will then describe the Lie algebra of $\mathcal{A}$, it is a locally free sheaf on S. As S is affine, we only need to consider $\mathrm{Lie}\mathcal{A}$ of global sections, let $\mathfrak{a} = \mathrm{Hom}(\Gamma, k)$ be the Lie algebra of A

Lemma 3.8. *There is a canonical isomorphism $Lie\mathcal{A} = Mor^W(S', Lie\mathcal{A}) = (k[S'] \otimes_k \mathfrak{a})^W$.*

Next we assume that we have an W-isomorphism $S' \otimes_{\mathbb{Z}} k$, we may identify S' with $\mathfrak{a}^*$, and we can define the module of Kahler differentials

$$\Omega(S') = k[S'] \otimes_k \mathfrak{a} = \mathrm{Lie}_{S'}(A \times S') \tag{3.1}$$

Theorem 3.9. *Assume $S' = \Gamma \otimes_{\mathbb{Z}} k$, then the equality* (3.1) *induces an isomorphism $\Omega(S) = Lie\mathcal{A}$.*

Example 3.10. Let X be a smooth G-variety, then the little Weyl group W_X acts on $\mathfrak{a}^*$ and also on $\chi(X)$, so we can form $\mathcal{A}_X$.

We introduce the notion of minimal root system attached to a pair (W, Γ)

Definition 3.11. The *minimal root system* $\Lambda = \Lambda(W, \Gamma) \subset \Gamma$ is the set of generators of $R_w := \{\gamma \in \Gamma \| \ \omega\gamma = -\gamma\}$ where $\omega \in W$ runs through all reflections.

It can verified that $\Phi = \mathbb{Z}\Lambda$ is indeed a root system and whose Weyl group is W.

3.2. **Integration of Lie algebra actions.** Let S be an affine variety, $\mathcal{A} \to S$ a smooth group scheme with *connected fibers*, and $L_{\mathcal{A}} = \text{Lie } \mathcal{A}$ the Lie algebra of $\mathcal{A}/S$ considered as a $k[S]$-module, we assume for every $s \in S$ and $\alpha \in \mathcal{A}_s^0$, there is a rational section $a : S \to \mathcal{A}^0$ with $a(s) = \alpha$. We will assume $\mathcal{A}$ is a group scheme defined by restriction of scalars from previous section.

Let $X \to S$ be an S-variety equipped with a Lie algebra homomorphism of $L_{\mathcal{A}}$ into the Lie algebra of global vector fields $\mathscr{T}(X/S)$, any group action $\mu : \mathcal{A} \times_S X \to X$ will induce a homomorphism like this.

We can define some universal S-scheme on which $\mathcal{A}$ acts: let $\mathfrak{X}$ be the set of all local rings $\mathcal{P} \subset k(X)$ satisfying the properties

- The field of fractions of $\mathcal{P}$ is $k(X)$.
- $\mathcal{P}$ is the localization of a finitely generated subalgebra of a prime ideal.
- $k[S] \subseteq \mathcal{P}$.

then $\mathfrak{X}$ is a S-scheme. We define $\mathfrak{X}_0 \subset \mathfrak{X}$ to be the subset of those local algebras which are $L_{\mathcal{A}}$-stable. $\mathfrak{X}_0$ is an open scheme and there is an action of $L_{\mathcal{A}}$ on $\mathfrak{X}_0$

Theorem 3.12. *There is a unique morphism $\mu : \mathcal{A} \times_S \mathfrak{X}_0 \to \mathfrak{X}_0$ which is a group scheme action and which induces the action of $L_{\mathcal{A}}$.*

3.3. **Automorphisms and root system.** Let X be a smooth variety with G-action, consider the cotangent bundle $\pi : T^*X \to X$, and the G-action induces the moment map

$$\Phi : T^*X \longrightarrow \mathfrak{g}^* : \ \alpha \mapsto l_\alpha$$

where $l_\alpha(\xi) = \alpha(\xi_{\pi(\alpha)})$. T^*X carries a symplectic structure ω, and each function f on T^*X induces a Hamiltonian vector field H_f, this defines a Poisson product $\{f, g\} = \omega(H_f, H_g)$. The moment map induces $\Phi^* : k[\mathfrak{g}^*] \to k[T^*X]$, and this is a Poisson morphism, we denote its image by R_0.

The Poisson center of R_0 is the algebra R_0^G of invariants. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra, by the Chevalley restriction theorem, we have an isomorphism $k[\mathfrak{g}^*]^G \cong k[\mathfrak{t}^*]^W = k[\mathfrak{t}^*/W]$, for W the Weyl group of G. We get a morphism $\Psi : T^*X \to \mathfrak{t}^*/W$, and R_0^G is the image of Ψ^*.

Definition 3.13. The elements of R_0 are called collective Hamiltonians and R_0^G the invariant collective Hamiltonians.

The problem that Knop wanted to solve is roughly whether there is a commutative algebraic group action on T^*X which integrates the Hamiltonian vector fields for R_0^G. More precisely, for $s \in \mathfrak{t}^*/W$, $f \in k[\mathfrak{t}^*/W]$ and $f_0 = f \circ \Psi \in R_0^G$, then H_{f_0} is parallel to the fiber $T_s^* = \Psi^{-1}(s)$, we get a Lie algebra homomorphism $\Omega_s(\mathfrak{t}^*/W) \to \mathscr{T}(T_s^*)$ and the problem is whether there is a group $\mathcal{A}_s$ integrating the Lie algebra action.

Actually we want to integrate an algebra which is a bit larger than R_0^G, we let R be the integral closure of R_0 inside $k[T^*X]$. With $L_X := \text{Spec } R^G$, we get a morphism $T^*X \to L_X$.

Theorem 3.14. *Let X be a smooth G-variety, then there is a finite reflection group W_X acting on the vector space $\mathfrak{a}_X^*$ and a W_X-stable lattice $\chi(X) \subseteq \mathfrak{a}_X^*$ such that for the group scheme $\mathcal{A}_X^0 = \mathcal{A}(W_X, \mathfrak{a}_X^*, \chi(X))^0$ the following holds*

- *There is an identification $\mathfrak{a}_X^*/W_X = L_X$.*
- *There is an action of $\mathcal{A}_X^0$ on T^*X over L_X.*
- *There is a commutative diagram*

$$\begin{array}{ccc} \Omega(L_X) & \xrightarrow{\Psi^*} & \Omega(T^*X) \\ \cong \big\downarrow 1 & & \cong \big\downarrow 2 \\ \textit{Lie } \mathcal{A}_X^0 & \longrightarrow & \mathscr{T}(T^*X) \end{array}$$

where the arrow 1 is the homomorphism from theorem 4.4, the arrow is the identification via the symplectic structure of T^*X and the bottom arrow is induced by the $\mathcal{A}_X^0$-action.

For $f \in k(X)^{(B)}$, let $\chi_f \in \chi(B)$ be its character, we will define $\chi(X)$ as the image of $k(X)^{(B)}$ in $\chi(B)$. Let A_X be the torus with character lattice $\chi(X)$, and $\mathfrak{a}_X^* = \chi(X) \otimes k$, the projection $T \to A_X$ induces $\mathfrak{a}_X^* \hookrightarrow \mathfrak{t}^*$, the main result of [Kno90] shows that there is a finite reflection group W_X acting on $\mathfrak{a}_X^*$ and a canonical isomorphism $\mathfrak{a}_X^*/W_X \cong L_X$.

The proof is reduced to the homogeneous case, and then the theory from [Kno94] is used.

Let X be a normal variety, then we can apply the previous theory to study the group of central automorphisms of X

$$\mathfrak{U}(X) = \{\varphi \in \mathrm{Aut}^G(X) \mid \varphi(f) \in k^* f \text{ for all } f \in k(X)^{(B)}\}$$

when X is spherical, this is the full automorphism group. Let $D \in \mathscr{D}$ be a color, it will be called underdetermined if there is a different color D' such that the restrictions of valuations $v_D,\ v_{D'}$ to $k(X)^B$ coincide.

Example 3.15. $X = SL_2/T$, for T the maximal torus, there are two undetermined colors, we have $\mathfrak{U}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and it interchanges the two undetermined colors.

Let X be a normal G-variety, there is an open G-stable subset X_0 of X such that $\mathfrak{U}(X_0) = \mathfrak{U}(X_1)$ for every open G-stable subset X_1 of X_0. We will denote this group by $\mathfrak{U}_X$.

Theorem 3.16. *Let X be a G-variety, then there is a unique homomorphism*

$$\lambda : \mathfrak{U}_X \to A_X \quad \varphi \mapsto \lambda_\varphi$$

such that $\varphi(f) = \lambda_\varphi(\chi_f) f$ for all $f \in k(X)^{(B)}$. This homomorphism is injective and if X is normal its image is closed.

Knop showed further that $\mathfrak{U}(X)$ belongs to the center of $\mathrm{Aut}^G(X)$ and if X is normal, then $\mathfrak{U}_X$ contains $\mathfrak{U}(X)$ as a closed subgroup of finite index.

We will now assume that $\mathfrak{U}(X) = \mathfrak{U}_X$, and let $\mathfrak{a}_X^1 \subseteq \mathfrak{a}_X^*$ be the points with trivial W_X-isotropy group. We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{U}_X & \xrightarrow{\subseteq} & \mathrm{Aut}^G(X) \\ \downarrow & & \downarrow \\ A_X & \longrightarrow & \mathrm{Aut}(T^*X \times_{L_X} \mathfrak{a}_X^1) \end{array}$$

there is an action of $\mathfrak{U}_X$ on both T^*X and $T^*X \times_{L_X} \mathfrak{a}_X^1$, this implies that the element $a \in A_X$ is W_X-invariant.

Definition 3.17. We define the root lattice Λ_X of X to be the kernel of $\chi(A_X) \to \chi(\mathfrak{U}_X)$ and the root system Φ_X of X to be the minimal root system attached to (Λ_X, W_X) as in definition 3.11.

From the definition we have $X^*(\mathfrak{U}_X) \cong X^*(A_X)/\Lambda_X$. For X quasi-affine, there is an easier construction of Φ_X.

Definition 3.18. We define the *spherical weights* of X to be

$$\chi^+ = \{\lambda \in X^*(A)^+ :\ V(\lambda)_{\bar{k}}^{H_{\bar{k}}} \neq 0\ \}$$

Let $k[X] = \oplus_{\lambda \in \chi^+} V(\lambda)$ be the isotypic decomposition of $k[X]$, and we define

$$\Lambda_X^+ :=\ \mathbb{Z}_{>0}\{\lambda + \mu - \nu :\ V(\nu) \subset V(\lambda) \cdot V(\mu)\ \}$$

Lemma 3.19. *Let X be quasi-affine, then we have*

- *Λ_X^+ is a free monoid, we denote Σ_X^n its set of free generators.*
- *$\Lambda_X = \mathbb{Z}\Lambda_X^+$.*
- *$\mathscr{V} = \{v \in Hom(\chi(X), \mathbb{Q}) \mid\ v(\Lambda_X^+) \geq 0\ \}$.*

The second property also implies that $\mathfrak{U}(X) = \mathfrak{U}_X$ for X quasi-affine. Here let's note that Σ_X and Σ_X^n is different in general, e.g. for X a wonderful variety, we have $\chi(X) = \langle \Sigma_X \rangle$. The precise relation between Σ_X and Σ_X^n is given by theorem 4.4.

When X is a symmetric variety, Φ_X is closely related to the classical restricted root system associated to symmetric varieties. So we assume that X is a symmetric variety and $X = G/H$ where G is semisimple and H is the fixed point of an involution θ. Let T be a θ-stable maximal torus and $\mathfrak{a} \subseteq \mathfrak{t}$ the (-1)-eigenspace of θ and $\rho : \mathfrak{t}^* \to \mathfrak{a}^*$ the restriction map, then $\Phi_X^r := \rho(\Delta)\backslash\{0\}$ is the *restricted root system for* X and it is well-known that Φ_X^r is indeed a root system though it might be non-reduced, that is it contains $\alpha \in \Delta_G$ also contains $\alpha/2$.

It can be shown that the root system Φ_X is compatible with the classical construction

Theorem 3.20. *Let X be a symmetric variety, then Φ_X is the reduced root system associated to $2\Phi_X^r$.*

Proof. There is a Borel subgroup B contains T and such that BH is dense in G, this implies $A_X \cong T/T\cap H$, from $T \cap H = T^\theta$, we get $\chi(X) = (1-\theta)\chi(T)$. We may assume G is of adjoint type, then $\chi(T)$ is generated by Δ, as $\frac{1}{2}(1-\theta)$ is the projection to $\mathfrak{a}^*$, we conclude $\chi(X)$ is the root lattice of $2\Phi_X^r$. Also $\chi(X)$ is the root lattice of Σ_X as $N_G(H) = H$, furthermore, it is known that Φ_X and Φ_X^r have the same Weyl group, so we conclude Φ_X is the reduced root system of $2\Phi_X^r$. □

Example 3.21. The restricted root system of the symmetric variety $GL_{2n}/S(GL_n \times GL_n)$ is of type C_n, the spherical root system is of type B_n as the reduced root system associated with $2C_n$ root system.

The restricted root system of the symmetric variety $GL_n/S(GL_m \times GL_{n-m})$ is of type BC_m, the spherical root system is of type B_m.

Definition 3.22. We define $\mathfrak{U}_X^\# \subset \mathfrak{U}_X$ to be the subgroup of automorphisms that stabilize every B-stable divisor of X.

Theorem 3.23. *Let X be a normal G-variety, then there is a root system $\Phi_X^\# \subseteq \chi(X)$ with simple roots $\Sigma_X^\#$ such that $\mathfrak{U}_X^\# = \cap_{\alpha\in\Sigma_X^\#} ker_{A_X}\alpha$.*

Furthermore, we can show that (this is essentially the fact that every spherical roots of type (1) is doubled)

Corollary 3.24. *Assume X is a G-spherical variety, then we have*

$$1 \longrightarrow \mathfrak{U}_X^\# \longrightarrow \mathfrak{U}_X \longrightarrow Aut_\Omega(\mathscr{D}) \longrightarrow 1$$

and $\mathfrak{U}_X^\# \cong X^*(A_X)/\langle\Sigma_X^\#\rangle$, $\mathfrak{U}_X = X^*(A_X)/\Lambda_X = X^*(A_X)/\langle\Sigma_X^n\rangle$.

4. Result of Losev

In this section, we will assume k is algebraically closed, and we will summarize the main result of [Los09].

Losev clarifies the distinction between Σ_X and Σ_X^n, and he introduced the notion of *distinguished spherical roots.*

Definition 4.1. Following Losev, we will call roots of type $(1), (2), (3)$ the set of distinguished roots.

We introduce the following families of spherical subgroups related to the spherical subgroup H, whose combinatorial invariants can be recovered from those of H.

Definition 4.2. We define $\mathcal{H}_H$ to be the set of all algebraic subgroups $\widetilde{H}$ of G such that $H \subset \widetilde{H}$ and $\widetilde{H}/H$ is connected.

We also define $\underline{\mathcal{H}}_H = \{\widetilde{H} \in \mathcal{H} | \ R_u(H) \subset R_u(\widetilde{H}),\ \widetilde{H}/R_u(\widetilde{H}) = H/R_u(H),\ R_u(\tilde{\mathfrak{h}})/R_u(\mathfrak{h})$ is an irreducible H-module$\}$.

we note that $\mathcal{H}_H$ can be described in terms of the so called *colored subspace.*

Given X_1, X_2 two spherical varieties, we write $\mathscr{D}(X_1) = \mathscr{D}(X_2)$ if there is a bijection $\psi : \mathscr{D}(X_1) \to \mathscr{D}(X_2)$, such that $G_D = G_{\psi(D)}$, $\rho(D) = \rho(\psi(D))$, here $G_D = \{g \in D \mid \ gD = D\}$.

Theorem 4.3. *Let H_1, H_2 be two spherical subgroups, $X_1 = G/H_1$, $X_2 = G/H_2$, if $\chi(X_1) = \chi(X_2)$, $\mathscr{V}_{X_1} = \mathscr{V}_{X_2}$, $\mathscr{D}(X_1) = \mathscr{D}(X_2)$, then H_1, H_2 are G-conjugate.*

The following theorem studies the relation between Σ_X and Σ_X^n.

Theorem 4.4. *Assume X is an affine spherical variety, then Σ_X^n is obtained from Σ_X by replacing spherical roots α of type $(1), (2), (3), (4)$ by 2α.*

Furthermore, it can be shown that any bijection $\psi : \mathscr{D}(X_1) \to \mathscr{D}(X_2)$ is induced by some element of $\mathfrak{U}_X$.

We have the following functorial properties between distinguished roots

Lemma 4.5. *Let X_1, X_2 be two spherical G-varieties and let $\varphi :\ X_1 \to X_2$ be a dominant G-equivariant morphism, then $\Sigma_{X_1}^i \cap \Sigma_{X_2} \subseteq \Sigma_{X_2}^i$ for $i = 1, 2, 3$. If φ is generically etale, then we have the equalities hold.*

The distinguished roots also behave well under parabolic induction

Proposition 4.6. *Let X be a spherical G-variety of the form $G \times_H V$, where H is a reductive subgroup of G and V is an H-module, then $\Sigma_X^i \subset \Sigma_{G/H}^i$ for $i = 1, 2$.*

Recall that $\mathfrak{U}_X$ is the image of $\mathrm{Aut}^G(X)$ inside A_X.

Definition 4.7. We say that $\varphi \in \mathfrak{U}_X$ doubles $\alpha \in \Sigma_X^{dist}$ if $\langle \alpha, \varphi \rangle = -1$. The set of all $\varphi \in \mathfrak{U}_X$ doubling α is denoted by $\mathfrak{U}_X(\alpha)$.

Lemma 4.8. *Theorem 4.4 is equivalent to the statement*

$$(*) \qquad \mathfrak{U}_X(\alpha) \neq \emptyset \text{ for any } \alpha \in \Sigma_X^{dist}$$

Proof. We may assume $X = G/H$, since $\overline{\Sigma}_X = \Sigma_{G/N_G(H)}$, theorem 4.4 implies the statement.

From the table of Wasserman, it is known that $\overline{\Sigma}_X = \Sigma_1 \sqcup 2\Sigma_2$, for some partition of $\Sigma_X = \Sigma_1 \sqcup \Sigma_2$ with $\Sigma_X \cap \Lambda_G \backslash \Sigma_X^{dist} \subset \Sigma_1$, if (4.1) holds, then $\Sigma_X^{dist} \subset \Sigma_2$, since the image of $Z(G)$ in $\mathrm{Aut}(X)$ belongs to $\mathrm{Aut}^G(X)$, we have $\Sigma_X \backslash \Lambda_G \subset \Sigma_2$. □

The following shows that the property $(*)$ behaves well under the quotient map $G/H^\circ \to G/H$.

Lemma 4.9. *We have the following assertions*

- *Let $\alpha \in \Sigma_X^{dist}$, then $\mathfrak{U}_{G/H^\circ}(\alpha) \neq \emptyset$ is equivalent to $\mathfrak{U}_{G/H}(\alpha) \neq \emptyset$.*
- *$(*)$ holds for G/H whenever it holds for $G/N_G(H)$.*

Lemma 4.10. *Let X be an arbitrary smooth spherical G-variety, then there is an epimorphism $Pic(X) \to \mathbb{Z}^{\Sigma_X^1}$.*

This lemma, in the setting $N_G(H)$ acts on $\mathfrak{z}(\mathfrak{h})$ without nonzero vectors will imply that $\#\mathrm{Pic}(G/N_G(H)) < \infty$, hence $\Sigma_{G/N_G(H)}^1 = \emptyset$.

The proof of the following lemma can be reduced to the G_2/SL_3 case.

Lemma 4.11. *Let $X = G/H$ and let $\alpha \in \Sigma_X^3$, then $\mathfrak{U}_X(\alpha) \neq \emptyset$.*

We can now sketch a proof of theorem 4.4 in the case of smooth affine spherical varieties.

The assertion (1) proves theorem 4.4 for some spherical affine homogeneous spaces and the assertion (2) is an auxiliary result in the proof of theorem 4.3.

Proposition 4.12. *Let H be a reductive subgroup of G such that $N_G(H)$ is not contained in a proper parabolic subgroup of G, then*

(1) the condition $(*)$ holds for $X = G/H$.
(2) $g^2 \in N_G(H)^\circ Z(G)$ for any $g \in N_G(H)$.

For the proof of (1), we may assume $N_G(H)^\circ = H$ by 4.9, and the reductive spherical subgroups $H \subset G$ with $N_G(H)^\circ = H$ are classified in [Bri87] and [Mik87]. Concerning the first type of the distinguished spherical roots, we have $\Sigma_{G/N_G(H)}^1 = \emptyset$ from the discussion after 4.10, and hence $\mathfrak{U}_X(\alpha) \neq \emptyset$ for any $\alpha \in \Sigma_X^1$. For the distinguished spherical root of type 3, (1) is proved in 4.9.

For the proof of (2), the symmetric variety case can be checked directly by hand, and in the case when G/H is not symmetric, if G is not simple, we have $N_G(H) = H$, if G is simple, there are only finitely many such pairs, one can check case by case that $N_G(H)/H \cong \mathbb{Z}_2$ or 1, hence $g^2 \in N_G(H)^\circ Z(G)$ holds.

We now sketch the proof for theorem 4.4: using the previous result 4.9, we may reduce to the case $H = N_G(H)$ and from 4.12, we may assume H is contained in a proper parabolic. Now we want to show that there are no distinguished spherical roots. This follows from the following propositions: if there is a distinguished root $\alpha \in \Sigma_{G/H}$, then we show that there exists $\widetilde{H} \in \mathcal{H}_H$ with $\alpha \in \Sigma_{G/\widetilde{H}}$, but the existence of such $\widetilde{H}$ contradicts the inductive assumption.

Proposition 4.13. *Let $\alpha \in \Sigma_X^i$ for $i = 1, 2$, suppose H is contained in some proper parabolic subgroup of G, then we have $\alpha \in \Sigma_{G/\widetilde{H}}$, for some $\widetilde{H} \in \underline{\mathcal{H}}_H$.*

Proposition 4.14. *Let $\alpha \in \Sigma_X^i$ for $i = 1, 2$, then there is no $\widetilde{H} \in \underline{\mathcal{H}}_H$ such that $\alpha \in \Sigma_{G/\widetilde{H}}$.*

5. Calculation of the automorphism group

In this section, we let $X = G/H$ be a spherical variety over a field k of characteristic 0, G a quasisplit connected reductive group over k. Then there is a k-structure on $\mathcal{A}_X = N_G(H)/H$ induced from $\mathcal{A}_X \hookrightarrow A_X$, and A_X has a k-torus structure. We use the previous result of Knop and Losev to give a combinatorial description of the automorphism group $\mathcal{A}_X := \mathrm{Aut}^G(G/H)$, for $a \in \mathcal{A}_X$ and $\lambda \in \chi$, the G-equivariant automorphism a preserves the one-dimensional subspace $k(G/H)^B_\lambda$, it acts on this space by a scalar $d_{a,\lambda} \in k^\times$. We obtain a homomorphism in this way

$$\iota : \mathcal{A}_X \to \mathrm{Hom}(\chi, k^\times),\ a \mapsto (\lambda \mapsto d_{a,\lambda})$$

Knop proved that the homomorphism ι is injective and its image is closed in $A_X(k)$, so its image corresponds to a lattice $\Lambda_X \subset \chi$, so $\mathcal{A}$ is the group of k-points of a group A of multiplicative type over k.

We let $\Sigma_X^\# \subset \Sigma_X$ be the subset of spherical roots obtained from Σ_X by replacing γ with 2γ for $\gamma \in \Sigma_X^2, \Sigma_X^3, \Sigma_X^4$. Recall that Σ_X^n is obtained from Σ_X by replacing γ with 2γ for all $\gamma \in \Sigma_X^i,\ i = 1, 2, 3, 4$. Let's note that Σ_X^n is obtained from $\Sigma_X^\#$ by doubling α for $\alpha \in \Sigma_X^1$ by definition.

Proposition 5.1. *We have $A = \langle \Sigma_X^n \rangle^\perp$, and hence $X^*(A) = \chi / \langle \Sigma_X^n \rangle$.*

This is exactly the content of theorem 4.4.

From 5.1, we have

$$A(k) = \mathrm{Hom}(\chi/\langle \Sigma_X^n \rangle, k^\times)$$

let $\gamma \in \Sigma_X^1 \subset \chi$, then $\gamma \notin \langle \Sigma^N \rangle$, but $2\gamma \in \Sigma_X^n$, we have $[\gamma]$ as an element of $\chi/\langle \Sigma_X^n \rangle$ of order 2, and it defines a homomorphism

$$A(k) = \mathrm{Hom}(\chi/\langle \Sigma_X^n \rangle, k^\times) \to \mathrm{Hom}(\langle \gamma \rangle / \langle 2\gamma \rangle, k^\times) = \{\pm 1\},\ a \mapsto a(\gamma)$$

for $a(f_\gamma) = a(\gamma) f_\gamma$ with $f_\gamma \in k(X)^B_\gamma$.

Proposition 5.2. *For $\gamma \in \Sigma_X^1$, an automorphism $a \in A(k) = Aut^G(G/H)$ swaps D_γ^+ and D_γ^- if and only if $a(\gamma) = -1$.*

the proof of this proposition is contained in Losev's remark of the paper [Los09] after his definition of $\mathfrak{U}_X(\alpha)$.

$\mathcal{A}_X$ is the k-points of $A = N_G(H)/H$, since $\mathcal{A}$ acts on $\mathscr{D}$, we have a homomorphism $\mathcal{A} \to \mathrm{Aut}(\mathscr{D})$, from Losev's result $\mathfrak{U}_X(\alpha) \in \alpha$, we see this homomorphism is surjective and hence we have the exact sequence

$$1 \longrightarrow \mathcal{A}_X^\# \longrightarrow \mathcal{A}_X \longrightarrow \mathrm{Aut}(\mathscr{D}) \longrightarrow 1$$

$\mathcal{A}_X^\#$ is the group of k-points of an algebraic subgroup $A^\#$ of A, and $A^\# = \overline{H}/H \subset N_G(H)/H = A$, here $\overline{H}$ is the spherical closure of H.

Proposition 5.3. *We have $A^\# = \langle \Sigma_X^\# \rangle^\perp$, and $X^*(A^\#) = \chi / \langle \Sigma_X^\# \rangle$.*

Proof. By proposition 5.2, we have $A = \langle \Sigma_X^n \rangle^\perp$, and from the definition, $a \in A(k) = \mathrm{Aut}^G(G/H)$ is contained in $A^\#(k)$ if and only if a fixes D_γ^+ and D_γ^- for all $\gamma \in \Sigma_X^1$, hence from proposition 5.2, this holds if and only if $a(\gamma) = 1$ for all $\gamma \in \Sigma^{(2)}$, since $\langle \Sigma_X^n \rangle + \langle \Sigma_X^1 \rangle = \langle \Sigma_X^\# \rangle$, so we conclude $A^\# = \langle \Sigma_X^\# \rangle^\perp$. ☐

The group A here corresponds to the $\mathfrak{U}_X$ and $A^\#$ corresponds to $\mathfrak{U}_X^\#$ in Knop's paper.

Example 5.4. We will denote α the simple root of PGL_2 under the upper triangular Borel subgroup.

For the spherical variety $X = PGL_2/T$, with T the maximal torus, there are two colors D_1, D_2 with same valuation. We have T is a spherically closed subgroup, as we have $\mathcal{A}_X^\# = 1$, and $\mathcal{A}_X \cong \mathbb{Z}/2\mathbb{Z}$, the nontrivial elements permute the two colors D_1, D_2.

Now we calculate $\mathcal{A}_X$ using propositions 5.2 and 5.3. X is a wonderful variety, so $\chi = \Sigma_X = \mathbb{Z}\alpha$. Σ_X^n is obtained from Σ_X by doubling α, so $\mathbb{Z}\Sigma_X^n = 2\mathbb{Z}\alpha$, hence $\mathcal{A}_X \cong \chi/\mathbb{Z}\Sigma_X^n \cong \mathbb{Z}\alpha/2\mathbb{Z}\alpha = \mathbb{Z}/2\mathbb{Z}$, Σ_X^{sc} is obtained from doubling spherical roots of type $(2), (3), (4)$, and there is no spherical roots of type $(2), (3), (4)$, so $\Sigma_X^\# = \Sigma_X$, hence $\mathcal{A}_X^\# = 1$.

Above all, we see that T is a wonderful, spherically closed subgroup of PGL_2 (as $\mathcal{A}_X^\# = 1$), and it is not self-normalizing (as $\mathcal{A}_X \neq 1$).

References

[Bri87] Michel Brion. Classification des espaces homogenes sphériques. *Compositio Mathematica*, 63(2):189–208, 1987.

[KK16] Friedrich Knop and Bernhard Krötz. Reductive group actions. *arXiv preprint arXiv:1604.01005*, 2016.

[Kno90] Friedrich Knop. Weylgruppe und momentabbildung. *Inventiones mathematicae*, 99(1):1–23, 1990.

[Kno94] Friedrich Knop. A harish-chandra homomorphism for reductive group actions. *Annals of Mathematics*, 140(2):253–288, 1994.

[Kno96] Friedrich Knop. Automorphisms, root systems, and compactifications of homogeneous varieties. *Journal of the American Mathematical Society*, 9(1):153–174, 1996.

[Los09] Ivan V Losev. Uniqueness property for spherical homogeneous spaces. 2009.

[Mik87] IV Mikityuk. On the integrability of invariant hamiltonian systems with homogeneous configuration spaces. *Mathematics of the USSR-Sbornik*, 57(2):527, 1987.

[SV17] Yiannis Sakellaridis and Akshay Venkatesh. *Periods and harmonic analysis on spherical varieties.* Société mathématique de France, 2017.