ASAI RANKIN-SELBERG INTEGRALS

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1. Introduction

This is a study note for Beuzart-Plessis' paper[BP21], BP partially completes the local Rankin-Selberg theory of Asai L-function and ϵ-factors as introduced by Flicker and Kable, in particular he establishes the relevant functional equation and prove the equality between Rankin-Selberg's and Artin's ϵ-factors in full generality. In this note, I will only focus on the p-adic case although BP's result applys in general.

Recall that the classical Rankin-Selberg local theory for tensor L-functions and ϵ-factors, the main results roughly say that we can define the local L-functions of pairs as the "greatest common divisor" of certain families of Zeta integrals and local ϵ-factors of pairs through certain functional equations satisfied by those families. Moreover, it is one of the characterizing properties of the local Langlands correspondence in the p-adic case these so defined local L-factors and ϵ-factors match with the Artin L and ϵ-factors in the Galois side.

Flicker has introduced in the non-Archimedean case a family of Zeta integrals that represent the Asai L-function of a given generic irreducible representation π of $GL_n(E)$ where E is a quadratic extension of a non-Archimedean field F and the Asai L-function is taken with respect to this extension. In particular, he was able to define a Rankin-Selberg type Asai L-function $L^{RS}(s,\pi,As)$ as the greatest common divisor of his family of Zeta integrals as well as a ϵ-factor $\epsilon^{RS}(s,\pi,As,\psi')$ where $\psi' :\ F \to S^1$ is a non-trivial character through the existence of a functional equation satisfied by the same Zeta integrals. Similar results have been obtained independently by Kable. To be more specific, let ψ be a nontrivial additive character of E which is trivial on F and let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π with respect to the corresponding standard character of the standard maximal unipotent subgroup $N_n(E)$ of $GL_n(E)$, the Zeta integrals defined by Flicker and Kable are associated to functions $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in C_c^\infty(F^n)$ and defined by

$$Z(s,W,\phi) = \int_{N_n(F)\backslash GL_n(F)} W(h)\phi(e_n h)|det\ h|_F^s\ dh$$

where s is a complex parameter and $e_n = (0,\cdots,0,1)$ and $|\cdot|_F$ the normalized absolute value on F.

By the work of Anandavardhanan-Rajan (for π square-integrable) and Martinge (for general π) the L-function $L^{RS}(s,\pi,As)$ matches Shahidi's Asai L-function $L^{Sh}(s,\pi,As)$ and hence by Henniart also the corresponding Artin L-function $L(s,\pi,As)$. In BP's paper, he completed the result on the equality between ϵ-factors and also worked out the Archimedean theory.

We now state the main result of the paper

Theorem 1.1. *Let $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in \mathcal{S}(F^n)$, then*

- *The integral defining $Z(s,W,\phi)$ is convergent when the real part of s is sufficiently large and moreover it extends to a meromorphic function on $\mathbb{C}$.*
- *We have the functional equation*

$$\begin{aligned}\frac{Z(1-s,\widetilde{W},\hat{\phi})}{L(1-s,\widetilde{\pi},As)} &= \omega_\pi(\tau)^{n-1}|\tau|_E^{\frac{n(n-1)}{2}(s-1/2)}\lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \\ &= \epsilon(s,\pi,As,\psi')\frac{Z(s,W,\phi)}{L(s,\pi,As)}\end{aligned}$$

where $L(s,\widetilde{\pi},As)$, $L(s,\pi,As)$ and $\epsilon(s,\pi,As,\psi')$ stand the Asai L and ϵ-factors of Artin.

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- *The function* $s \mapsto \frac{Z(s,W,\psi)}{L(s,\pi,As)}$ *is holomorphic. Moreover, if* π *is nearly tempered, for every* $s_0 \in \mathbb{C}$ *we can choose* $W \in \mathcal{W}(\pi,\psi)$ *and* $\phi \in \mathcal{S}(F^n)$ *such that this function does not vanish at* s_0*.*

2. Local Zeta integrals

2.1. **Definition and convergence of the local zeta integrals.** Let $\pi \in \mathrm{Irr}_{gen}(G_n(E))$, for every $W \in \mathcal{W}(\pi,\psi_n)$, $\phi \in \mathcal{S}(F^n)$ and $s \in \mathbb{C}$ we define, whenever convergent, a Zeta integral

$$Z(s,W,\phi) = \int_{N_n(F)\backslash G_n(F)} W(h)\phi(e_n h)|det\ h|_F^s\ dh$$

The next lemma shows that the above Zeta integrals at least convergent in some right half-plane

Lemma 2.1. *Let* $\mu \in \mathcal{A}^*$ *and* $\phi \in \mathcal{S}(F^n)$*, then for every* $W \in C_\mu(N_n(E)\backslash G_n(E),\psi_n)$*, the integral*

$$Z(s,W,\phi) = \int_{N_n(F)\backslash G_n(F)} W(h)\phi(e_n h)\ |det\ h|_F^s\ dh$$

is absolutely convergent for all $s \in \mathcal{H}_{>-2\ min\ \mu}$*, moreover, the function* $s \mapsto Z(s,W,\phi)$ *is holomorphic and bounded in vertical strips.*

2.2. **Local functional equation.**

2.2.1. *Local functional equation: The split case.* In this section, we assume that we are in the split case, $E = F \times F$, let $\pi \in \mathrm{Irr}_{gen}(G_n(E))$, then $\pi = \pi_1 \boxtimes \pi_2$ for some $\pi_1,\ \pi_2 \in \mathrm{Irr}(G_n(F))$, in the case where $\psi(x,y) = \psi'(x)\psi'(-y)$ for $(x,y) \in E$ and $W = W_1 \otimes W_2$ for some $W_1 \in \mathcal{W}(\pi_1,\psi'_n)$, $W_2 \in \mathcal{W}(\pi_2,\psi_n'^{-1})$, for $\phi \in \mathcal{S}(F^n)$, the Zeta integral belongs to a family studied by Jacquet-Piatetskii-Shapiro and Jacquet.

By their main results together with some of the characterizing properties of the local Langlands correspondence for GL_n, in this situation $Z(s,W,\phi)$ admits a meromorphic continuation to $\mathbb{C}$ satisfying the functional equation

$$\begin{aligned} Z(1-s,\widetilde{W},\hat{\phi}) &= \omega_{\pi_2}^{n-1}\gamma(s,\pi_1\times\pi_2,\psi')Z(s,W,\phi) \\ &= \omega_\pi(\tau)^{n-1}|\tau|_E^{\frac{n(n-1)}{2}(s-\frac{1}{2})}\lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}}\gamma(s,\pi,\mathrm{As},\psi')Z(s,W,\phi) \end{aligned}$$

Here we note that the γ factor is the γ-factor as defined in Tate's thesis.

Theorem 2.2. *Assume that* $E = F \times F$*, let* $\pi \in Irr_{gen}(G_n(E))$*, then for every* $W \in \mathcal{W}(\pi,\psi_n)$ *and* $\phi \in \mathcal{S}(F^n)$*, the function* $s \mapsto Z(s,W,\phi)$ *has a meromorphic extension to* $\mathbb{C}$ *and satisfying the functional equation*

$$Z(1-s,\widetilde{W},\hat{\phi}) = \omega_\pi(\tau)^{n-1}|\tau|_E^{\frac{n(n-1)}{2}(s-\frac{1}{2})}\lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}}\gamma(s,\pi,As,\psi')Z(s,W,\phi)$$

2.2.2. *Local functional equation: The inert case.* In this section, we will assume that E/F is a quadratic field extension, then we are going to state two theorems which are the main results of this paper.

Theorem 2.3. *Assume that* E/F *is a quadratic field extension, let* $\pi \in Irr_{gen}(G_n(E))$*, then for every* $W \in \mathcal{W}(\pi,\psi_n)$ *and* $\phi \in \mathcal{S}(F^n)$*, the function* $s \mapsto Z(s,W,\phi)$ *has a meromorphic extension to* $\mathbb{C}$ *and satisfies the functional equation*

$$Z(1-s,\widetilde{W},\hat{\phi}) = \omega_\pi(\tau)^{n-1}|\tau|_E^{\frac{n(n-1)}{2}(s-\frac{1}{2})}\lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}}\gamma(s,\pi,As,\psi')Z(s,W,\phi)$$

Theorem 2.4. *Assume that* E/F *is a quadratic field extension, let* $\pi \in Irr_{gen}(G_n(E))$*, then for every* $W \in \mathcal{W}(\pi,\psi_n)$ *and* $\phi \in \mathcal{S}(F^n)$*, the function*

$$s \mapsto \frac{Z(s,W,\phi)}{L(s,\pi,As)}$$

is holomorphic and of finite order in vertical strips.

2.3. **Unramified computation.** In this section, we consider the case where F is a non-Archimedean local field and extension E/F is either inert or split. If π and ψ are unramified there is a unqiue $W \in \mathcal{W}(\pi, \psi_n)^{G_n(\mathcal{O}_E)}$ such that $W(1) = 1$.

Lemma 2.5. *Assume that F is non-Archimedean and $\pi \in Irr_{gen}(G_n(E))$, ψ and ψ' are all unramified, let $W \in \mathcal{W}(\pi, \psi_n)^{G_n(\mathcal{O}_E)}$ be normalized by $W(1) = 1$ and $\phi \in \mathcal{S}(F^n)$ be the characteristic function of $\mathcal{O}_F^n$, then for $Re(s) >> 1$, we have*

$$Z(s, W, \phi) = Vol(N_n(\mathcal{O}_F)\backslash G_n(\mathcal{O}_F))L(s, \pi, As)$$

Note that by Iwasawa decomposition, we have

$$Z(s, W, \phi) = \mathrm{Vol}(N_n(\mathcal{O}_F)\backslash G_n(\mathcal{O}_F)) \sum_{\lambda \in \mathbb{Z}^n} W(a(\lambda))\phi(\varpi_F^{\lambda_n} e_n)|det\ a(\lambda)|_F^s\ \delta_n(a(\lambda))^{-1}$$

then the rest follows from the Casselman-Shalika formula for the Whittaker function.

3. Global Zeta integrals and their functional equation

In this section we let k'/k be a quadratic extension of number fields, for every place v of k we denote by k_v the corresponding completion of k and set $k'_v = k_v \otimes_k k'$, also if v is non-Archimedean, we let $\mathcal{O}_v,\ \mathcal{O}_{k',v}$ be the ring of integers of k_v and k'_v respectively and $|\cdot|$ be the normalized absolute value on $\mathbb{A}$.

For ω a continuous character of Z_∞, we denote by $\mathcal{A}_{cusp}(Z_\infty G_n(k')\backslash G_n(\mathbb{A}_{k'}), \omega)$ the space of smooth functions $\varphi:\ G_n(k')\backslash G_n(\mathbb{A}_{k'}) \to\ \mathbb{C}$ with all their derivatives of moderate growth having central character ω $\varphi(zg) = \omega(z)\varphi(g)$ for every $(z, g) \in Z_\infty \times G_n(\mathbb{A}_{k'})$ and satisfying

$$\int_{N(k')\backslash N(\mathbb{A}_{k'})} \varphi(ug)\ du = 0$$

for every parabolic subgroup $P = MN$ of G_n.

Let Π be a cuspidal automorphic representation of $G_n(\mathbb{A}_{k'})$ and for every place v of k, W_v be a Whittaker function in $\mathcal{W}(\Pi_v, \Psi_{n,v})$ such that W_v is $G_n(\mathcal{O}_{k',v})$-invariant and satisfies $W_v(1) = 1$ for almost all places v, set

$$W = \prod_v\ W_v$$

then W is a well-defined function on $G_n(\mathbb{A}_{k'})$ satisfying $W(ug) = \Psi_n(u)W(g)$ for every $(u, g) \in N_n(\mathbb{A}_{k'}) \times G_n(\mathbb{A}_{k'})$. For every place v, choose a function $\phi_v \in S(k_v^n)$ such that $\phi_v = 1_{G_n(\mathcal{O}_{k',v})}$ for almost all places v and set $\phi = \prod_v\ \phi_v$. We define similarly $\hat{\phi} = \prod_v\ \hat{\phi}_v$ where the local Fourier transforms are defined with respect to the local components Ψ'_v of Ψ'. For $s \in \mathbb{C}$, we define whenever convergent, the following global analog of the Zeta integrals discussed in the previous sections:

$$Z(s, W, \phi) = \int_{N_n(\mathbb{A})\backslash G_n(\mathbb{A})} W(h)\phi(e_n h)|\mathrm{det}\ h|^s\ dh$$

Then we have the following result of Kable

Theorem 3.1. *When $Re(s)$ is sufficiently large, the integral defining $Z(s, W, \phi)$ is absolutely convergent and we have*

$$Z(s, W, \phi) = \prod_v\ Z(s, W_v, \phi_v)$$

moreover, the function $s \mapsto Z(s, W, \phi)$ has a meromorphic continuation to $\mathbb{C}$ that satisfies the functional equation

$$Z(s, W, \phi) = Z(1 - s, \widetilde{W}, \hat{\phi})$$

We also have the following globalization result, let $v_0,\ v_1$ be two distinct places of k with v_1 non-Archimedean, there is a natural topology on $\mathrm{Temp}(G_n(k'_{v_0}))$ of tempered representations of $G_n(k'_{v_0})$ with a topology. We have the following globalization result

Theorem 3.2. *Let U be an open subset of $Temp(G_n(k'_{v_0}))$, then there exists a cuspidal autormophic representation Π of $G_n(\mathbb{A}_{k'})$ such that $\Pi_{v_0} \in U$ and Π_v is unramified for every non-Archimedean place $v \notin \{v_0,\ v_1\}$.*

References

[BP21] Raphaël Beuzart-Plessis. Archimedean theory and ϵ-factors for the asai rankin-selberg integrals. In *Relative Trace Formulas*, pages 1–50. Springer, 2021.