ADAMS CONJECTURE

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1. Introduction

In this paper, we recall Sakellaridis' formulation of the Adams conjecture using the relative Langlands point of view, namely a Plancherel theoretic version of the Adams conjecture [Sak17].

2. Groups

Fix a sign $\epsilon = \pm 1$ and a field E of degree 1 or 2 over our base field k. We will work in the context of Kudla-Rallis' Siegel-Weil formula, thus we will consider the theta correspondence between an almost arbitrary group G_1 of an ϵ-hermitian form on a vector space V and the isometry group G_2 of an $-\epsilon$-hermitian form on a vector space W. When necessary, one of these groups will be replaced by its double metaplectic cover. We can interchange G_1 and G_2 and we will take G_2 to be the "smaller" of these groups.

More precisely, let k be a number filed, $E = k$ or a quadratic field extension, $\eta = \eta_{E/k}$ the quadratic idele class character of k attached to k to E by class field theory. The action of the Galois group of E/k will be denote by a $\bar{\cdot}$.

Fix a sign $\epsilon = \pm 1$ and consider a non-degenerate ϵ-hermitian space V, that is V is a vector space over E equipped with a non-degenerate ϵ-hermitian form

$$(v, \omega) = \epsilon\overline{(\omega, v)}$$

similarly, let W denote a non-degenerate $(-\epsilon)$-hermitian form space of dimension n.

We will denote by G_1 and G_2 respectively, certain central extensions of the isometry groups of V and W by $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$

	$G_1(V)$	$G_2(W)$	$d(n)$
$(E : k) = 2$	U_m	U_n	n
$\epsilon = -1$, n even	Sp_m	O_n	$n - 1$
$\epsilon = -1$, n odd	Mp_m	O_n	$n - 1$
$\epsilon = 1$, m even	O_m	Sp_n	$n + 1$
$\epsilon = 1$, m odd	O_m	Mp_n	$n + 1$

Here for a dual pair (G_1, G_2) as in the table 2, assuming that $m \geq d(n)$, let r denote the Witt index of V (the dimension of its maximal isotropic subspace)

- when $r = 0$ or $m - r > d(n)$ we are in the *convergent range*.
- the case $m < d(n)$ is in the *first term range*.
- the case $m = d(n)$ is the *boundary range*.
- the case $d(n) < m \leq 2d(n)$ is the *second term range*.

3. The L-group and Arthur SL_2 of Howe duality

We set $G_v = G_{1,v} \times G_{2,v}$ and ${}^LG = {}^LG_1 \times_{\mathrm{Gal}(\bar{k}/k)} {}^LG_2$. From now on we assume throughout by symmetry that $d(n) \leq m$, i.e. G_2 is the small group, we define *L-group of Howe duality* to be

$${}^LG_\omega := {}^LG_2$$

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endowed with a canonical morphism

$$^L G_\omega \times SL_2 \longrightarrow {}^L G$$

which is described as follows: to define the restriction of this map to $^L G_\omega$, it is enough to define two L-morphisms: $^L G_\omega \to {}^L G_1$ and $^L G_\omega \to {}^L G_2$. The latter is taken to be the identity and the former will be the natural morphism which will be described below, we then map SL_2 to the centralizer of the image of $^L G_\omega$ in the connected component of $^L G_1$.

For non-tempered representations of the small group, the "naive" version of the conjecture needs to be corrected in this case.

When V is unitary and the difference $m - n$ of the dimensions of V and W is odd, so exactly one of the two covers $\widetilde{GL}_m^n$ and $\widetilde{GL}_n^m$ is nonsplit, we need to map the connected component of $GL_n(\mathbb{C})$ into the top left block of $GL_m(\mathbb{C})$ and σ_n to the element

$$\begin{pmatrix} & I_n \\ I_{m-n} & \end{pmatrix} \sigma_m$$

Conjecture 3.1. *The theta lift to $G_{1,v}$ of a tempered representation π of $G_{2,v}$ with Langlands parameter ϕ, if non-zero, belongs to an Arthur packet with Arthur parameter*

$$\phi' : \mathcal{W}'_{k_v} \times SL_2 \longrightarrow {}^L G_2 \times SL_2 \longrightarrow {}^L G_1$$

here $\mathcal{W}'_{k_v}$ denotes the Weil group in the Archimedean case and Weil-Deligne group in the non-Archimedean case, the map $^L G_2 \times SL_2 \longrightarrow {}^L G_1$ is the map γ we defined before.

In all cases but the "base case" of the "going-up tower", it is immediate to see that the Langlands parameters given by Atobe-Gan are the ones of the main Langlands packet inside our desired Arthur packets. In the remaining cases, one needs to argue that the stated representations belong to our desired Arthur packet, as was done by Moeglin for symplectic-even orthogonal pairs.

4. A Plancherel-theoretic version of Adams' conjecture

Now consider the unitary oscillator representation ω_v at a place v as a genuine, unitary representation of the dual pair $G_v = G_{1,v} \times G_{2,v}$, throughout this paper, the "unitary dual" of these groups means the genuine unitary dual.

The abstract theory of the Plancherel formula tells us that there is a decomposition

$$\omega_v = \int_{\widehat{G}_v} \mathcal{H}_\pi \mu_v(\pi) \tag{4.1}$$

where μ_v is a measure on the unitary dual and for an irreducible unitary representation π of G_v, the unitary representation $\mathcal{H}_\pi$ is isomorphic to a sum of copies of π.

For the decomposition (4.1) to make sense, one needs to specify morphisms from a dense subspace ω_v^0 of ω_v to the Hilbert space $\mathcal{H}_\pi$ and the decomposition is essentially unique, we can show that in this case one can take $\omega_v^0 = \omega_v^\infty$. The decomposition is pointwise defined on ω_v^∞, assuming this, for μ-almost every irreducible $\pi = \pi_1 \otimes \pi_2$ of G_v, the morphism $\omega_v^\infty \to \mathcal{H}_\pi$ of the Plancherel decomposition has to factor through a π_v-isotypic quotient of ω_v^∞, the Howe duality theorem implies that π_1 and π_2 completely determine each other and the quotient is multiplicity-free, hence $\pi_1 = \theta(\pi_2)$, hence we have a Plancherel decomposition of the form

$$\omega_v = \int_{\widehat{G_{2,v}}} \theta(\pi_2) \hat{\otimes}\ \pi_2\ \mu_{2,v}(\pi_2)$$

where $\mu_{2,v}$ is some measure on the unitary dual of $G_{2,v}$.

We now formulate the following unitary variant of Adams' conjecture, it is the analog of the "relative local Langlands conjecture" for the L^2 space of a spherical variety. Recall that we are assuming that G_2 is the "small" group, i.e. $m \geq d(n)$.

Conjecture 4.1. *There is a direct integral decomposition*

$$\omega_v = \int_{[\phi]} \mathcal{H}_\phi \mu_{2,v}(\phi)$$

where

- $[\phi]$ *runs over the isomorphism classes of local tempered Langlands parameters into* ${}^LG_\omega = {}^LG_2$.
- $\mu_{2,v}$ *is the natural class of measures on the set of such Langlands parameters.*
- $\mathcal{H}_\phi$ *is isomorphic to a direct sum of irreducible representations belonging to the Arthur packet associated to the decomposition*

$$\mathcal{W}'_{k_v} \times SL_2 \longrightarrow {}^LG_\omega \times SL_2 \longrightarrow {}^LG_1 \times G_2$$

It is clear that this conjecture follows immediately from Adams' conjecture, once one knows that the Plancherel measure $\mu_{2,v}$ of the oscillator representation is absolutely continuous with respect to the Plancherel measure of the group $G_{2,v}$.

References

[Sak17] Yiannis Sakellaridis. Plancherel decomposition of Howe duality and Euler factorization of automorphic functionals. In *Representation Theory, Number Theory, and Invariant Theory: In Honor of Roger Howe on the Occasion of His 70th Birthday*, pages 545–585. Springer, 2017.