AN AFL FOR THE WHOLE HECKE ALGEBRA

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1. Introduction

A note taken by me on 03/13/2023 for Rapoport's talk "An AFL for the whole Hecke algebra" at MSRI.

2. FL and AFL

For $p \neq 2$, F/F_0 unramify quadratic extension, $n \geq 1$.

Fundamental lemma of Jacquet-Rallis: for W_0 a hermitian F/F_0-space, dim$= n+1$. $W_0^b = \langle u \rangle^\perp$, $||u_0|| = 1$. $G_{W_0} = U(W_0^b) \times U(W_0)$ algebraic action by $U(W_0^b) \times U(W_0^b)$. $G' = GL_n(F) \times GL_{n+1}(F)$.

<u>Fundamental lemma:</u> For matching elements (g, γ), we have

$$\mathcal{O}_g(1_{K^b \times K}) = \omega(\gamma)\mathcal{O}_\gamma(1_{K'_n \times K'_{n+1}})$$

- $\omega(\gamma)$: transfer factor.
- $\mathcal{O}_\gamma$: weighted orbital integral.

Long history: Jacquet-Rallis, Wei Zhang, Zhiwei Yun, Beuzart-Plessis.

There is a recent extension by Spencer-Leslie:

$$\mathcal{O}_g(\varphi) = \omega(\gamma)\mathcal{O}_\gamma(\varphi')$$

where $\varphi =$ image of $\varphi' \in BC : \mathscr{H}_{K'_n \times K'_{n+1}} \to \mathscr{H}_{K^b \times K}$.

The proof of the fundamental lemma reduces the equality to the unit element, the generalization of Spencer goes other way around.

Replace split W_0 by non-split W_1, then $W_1^b = \langle u_1 \rangle^\perp, ||u_1|| = 1$. $G_{W_1} = U(W_1^b) \times U(W_1)$.

<u>Arithmetic fundamental lemma:</u> For matching elements (g, γ)

$$2 \cdot \langle g\Delta, \Delta \rangle_{\mathcal{N}_{n,n+1}} logq = -\omega(\gamma) \cdot \partial\mathcal{O}_\gamma(1_{K'_n \times 1'_{K_{n+1}}})$$

here $\mathcal{N}_n =$ formal moduli space of $(X, \iota, \lambda, \rho)$ where

- $X =$ formal $\mathcal{O}_{F_0}$-module.
- $\iota : \mathcal{O}_F \longrightarrow End(X)$ such that $Lie(X)$ is of signature $(1, n-1)$.
- $\lambda =$ principal polarization, compatible with ι.
- $\rho =$ framing with $(\mathbb{X}, \iota_\mathbb{X}, \lambda_\mathbb{X})$.

this moduli space is represented by a formal scheme $\mathcal{N}_n$ formally smooth of relative dimension $n-1$ over $Spf(\mathcal{O}_{\breve{F}})$.

Δ is the graph of

$$\begin{aligned} \delta : \mathcal{N}_n &\longrightarrow \mathcal{N}_{n+1} \\ X &\longmapsto X \times \overline{\mathscr{E}} \end{aligned}$$

$\langle \ , \ \rangle =$ intersection product $= \chi(\mathcal{N}_{n,n+1}, \mathcal{O}_{g\Delta} \otimes^L \mathcal{O}_\Delta)$.

Long history: Wei Zhang, Mihatsch-Zhang, Zhiyu Zhang.

3. AFL for the whole Hecke algebra

The goal of today's talk: to present conjectural full AFL, which can be viewed an arithmetic version of Leslie's full fundamental lemma.

Conjecture:

$$2\langle \Delta, T(\varphi)\Delta\rangle_{\mathcal{N}_{n,n+1}} logq = -\omega(\gamma)\cdot \partial\mathcal{O}_{\gamma}(\varphi')$$

Theorem 3.1. *full AFL holds for $n = 1$.*

We have a map $\mathbb{Z}[X] \to Corr(\mathcal{N}_n, \mathcal{N}_n) : X \mapsto \tau_1$, $\mathbb{Z}[X] \otimes \mathbb{C} = \mathscr{H}(G_{W_0})$. The heart of the proof is that all the relevant maps between the local R-Z spaces are finite and flat.

Same procedure works in global case, e.g. for the modular curve.

for general n: Set $m = [\frac{n}{2}]$,

$$\mathcal{N}_n^{[t]} = \{(X, \iota, \lambda, \rho)|\ Ker(\lambda) \subset X(\varpi),\ |Ker\ \lambda| = q^{2t}\}$$

For $t' \leq t$,

$$\mathcal{N}^{[t,t']} = \{(X, X') \in \mathcal{N}^{[t]} \times \mathcal{N}^{[t']} + \text{lifting } \alpha : X^{[t]} \to X^{[t']}, \text{lift } \mathbb{X}^{[t]} \to \mathbb{X}^{[t']}, \alpha^*(\lambda') = \alpha\}$$

For $t' \geq t$, set $\mathcal{N}^{[t,t']} = {}^t\mathcal{N}^{[t',t]}$.

Proposition 3.2. *(Gatz, He, Rapo) Let $n \geq 2$, $t \neq t'$, consider the forgetful morphism*

$$\pi : \mathcal{N}^{[t,t']} \longrightarrow \mathcal{N}^{[t']}$$

then π is finite flat if and only if $t = 1, t' = 0$.

This is a serious problem, as in general

$$(\tau_t)_* \neq (\tau_t^+)_* \circ (\tau_t^-)_* : K(\mathcal{N}_n) \to K(\mathcal{N}_n)$$

We can define $\mathbb{T}_t$ as $(\tau_t^+)_* \circ (\tau_t^-)_*$. The problem is: does $\mathbb{T}_t$ commute with each other for varying t? If this holds, then as in the dim 1 case, we can define

$$\mathscr{H}_{K_n} = \mathbb{C}[x_1, \cdots, x_m] \longrightarrow End(K(\mathcal{N}_n))$$
$$X_t \longmapsto \mathbb{T}_t$$

For full AFL, have to replace $\mathcal{N}_n$ by $\mathcal{N}_{n,n+1}$

$$\mathscr{H}_{K^b\times K} = \mathscr{H}_{K^b} \otimes \mathscr{H}_K = \mathbb{C}[x_i] \otimes \mathbb{C}[y_i]$$